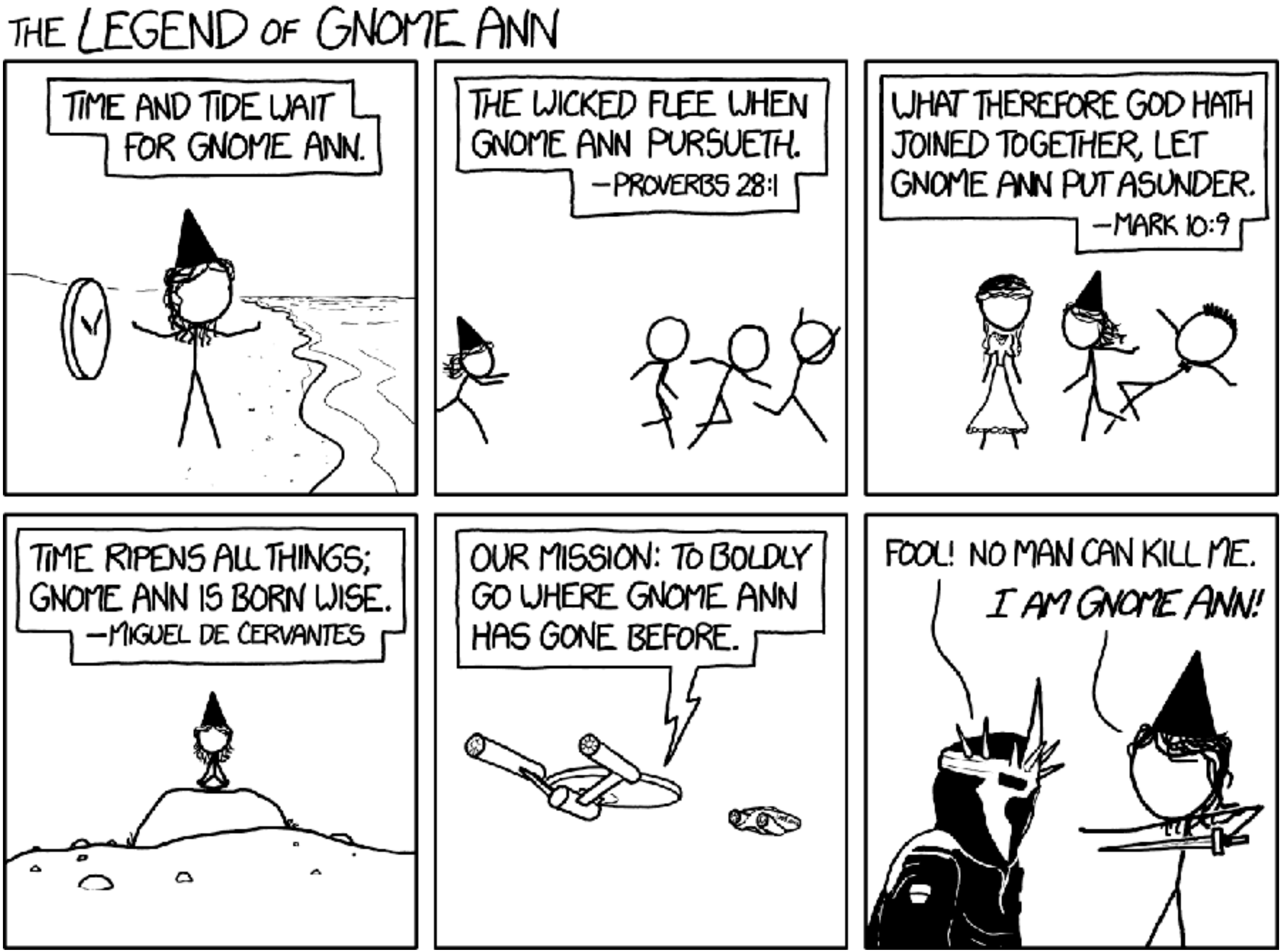
# LRA Interpolants from No Man's Land

Leonardo Alt, **Antti E. J. Hyvärinen**, and Natasha Sharygina
University of Lugano, Switzerland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TIME AND TIDE WAIT FOR GNOME ANN.
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—PROVERBS 28:1
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—MARK 10:9
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OUR MISSION: TO BOLDLY GO WHERE GNOME ANN HAS GONE BEFORE.
FOOL! NO MAN CAN KILL ME.
I AM GNOME ANN!

# Motivation

The goal: Finding the right proof

The tool: Make interpolation on LRA more flexible

The application: LRA for abstractions in software model checking

The keywords: SMT solving, function summaries, labeled interpolation systems

# Interpolants

Given two formulas $\mathrm{A}$ and $\mathrm{B}$ such that

$$A \wedge B \rightarrow \bot$$

an interpolant is a formula $I$ such that

$$\mathrm{Vars}(I) \subseteq \mathrm{Vars}(A) \cap \mathrm{Vars}(B)$$

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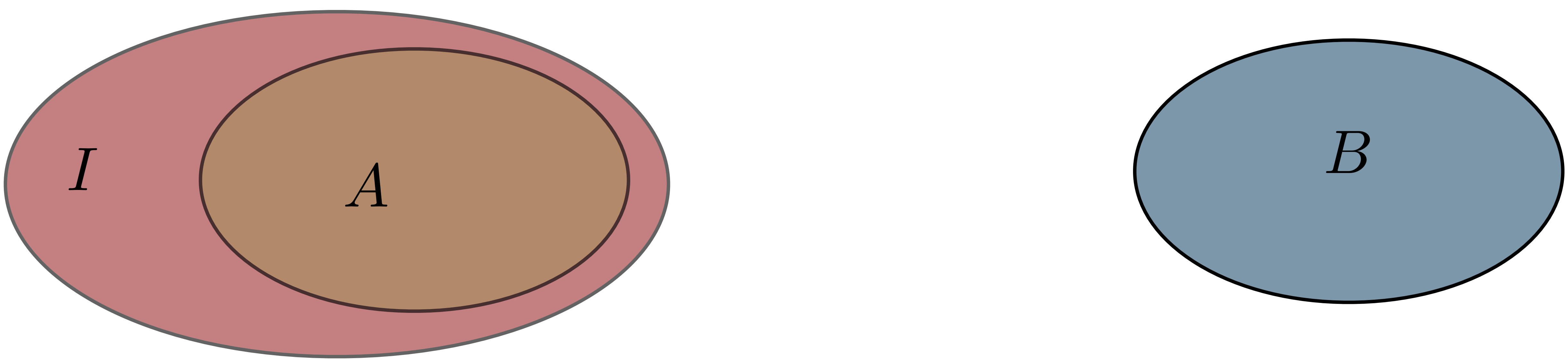

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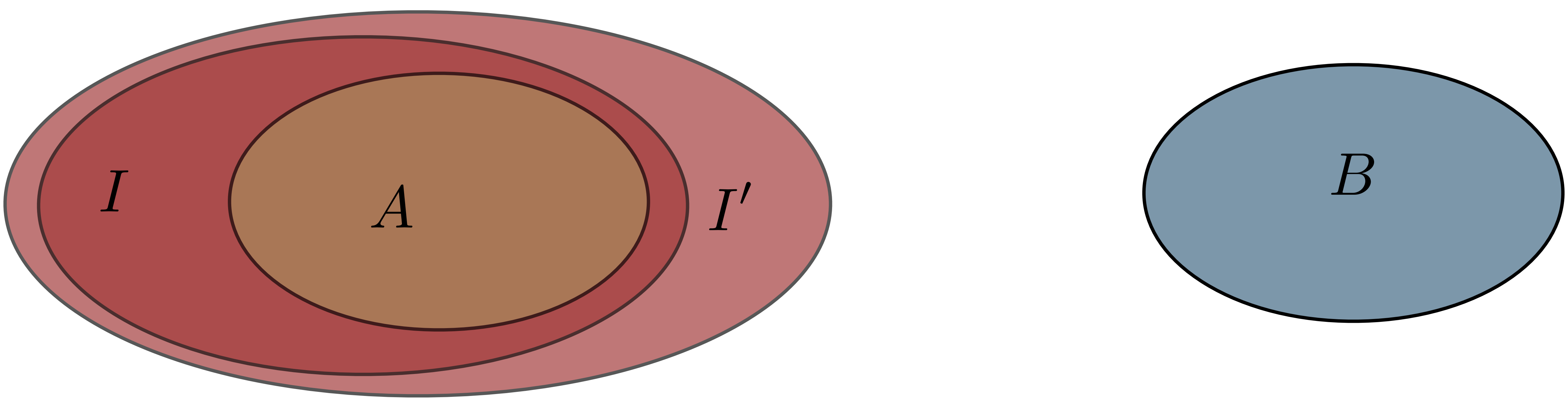

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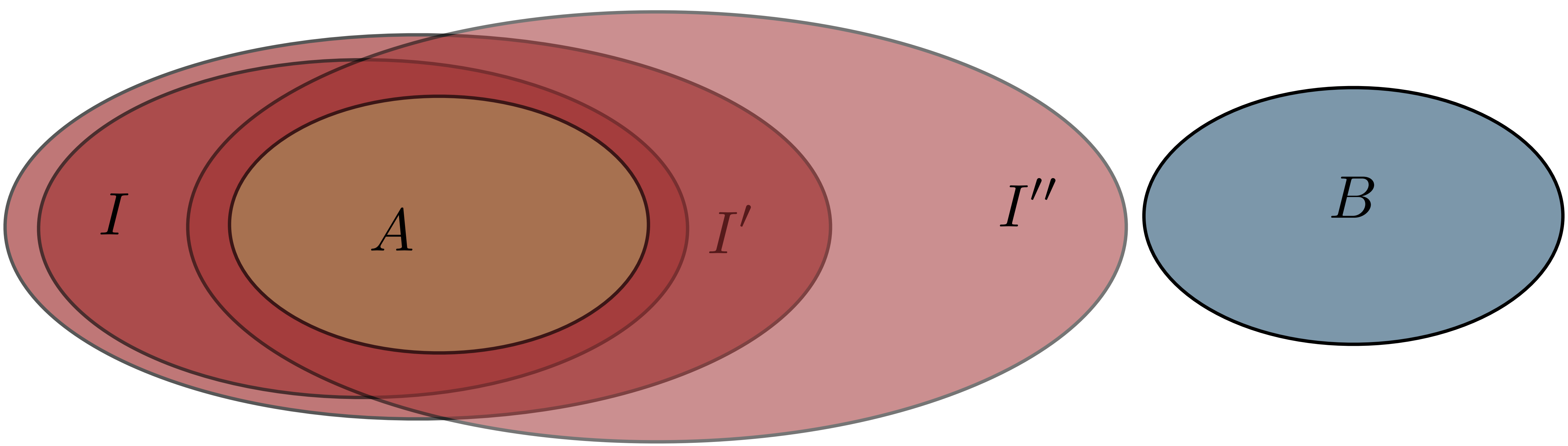

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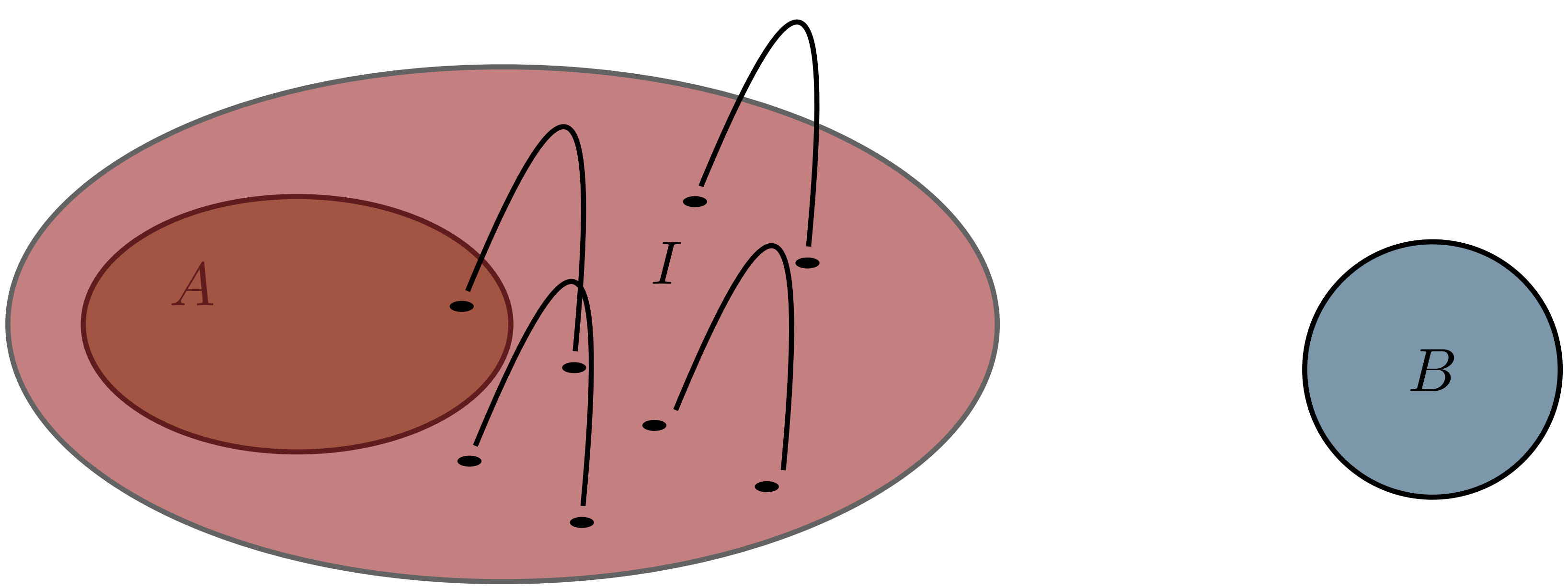
$$I(x) \wedge T(x, x') \rightarrow I(x')$$



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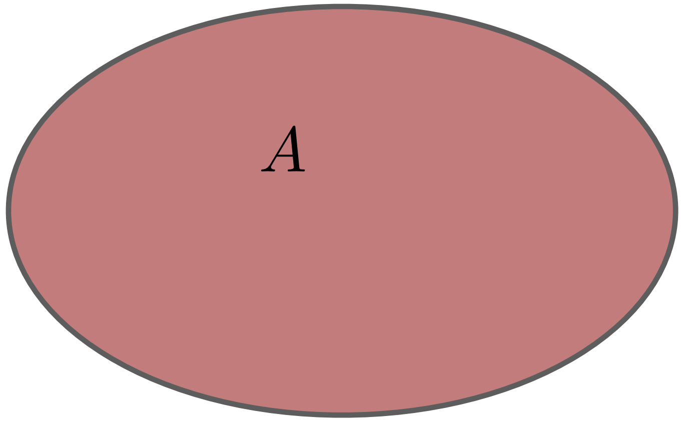

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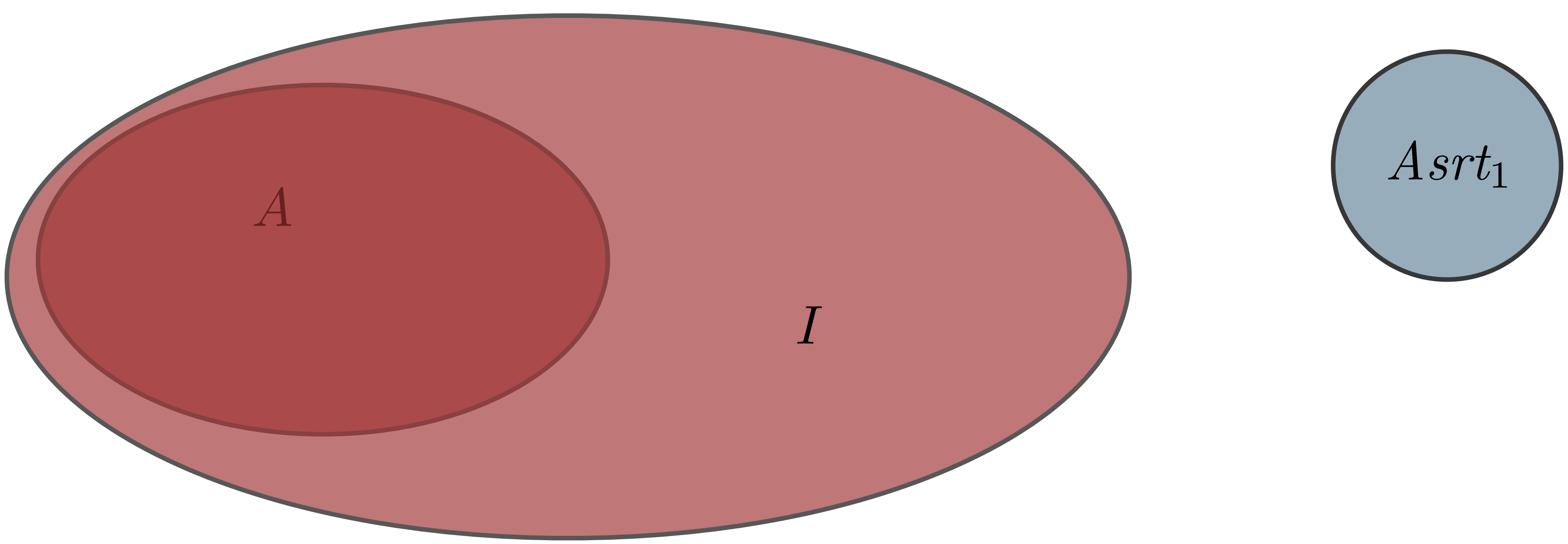

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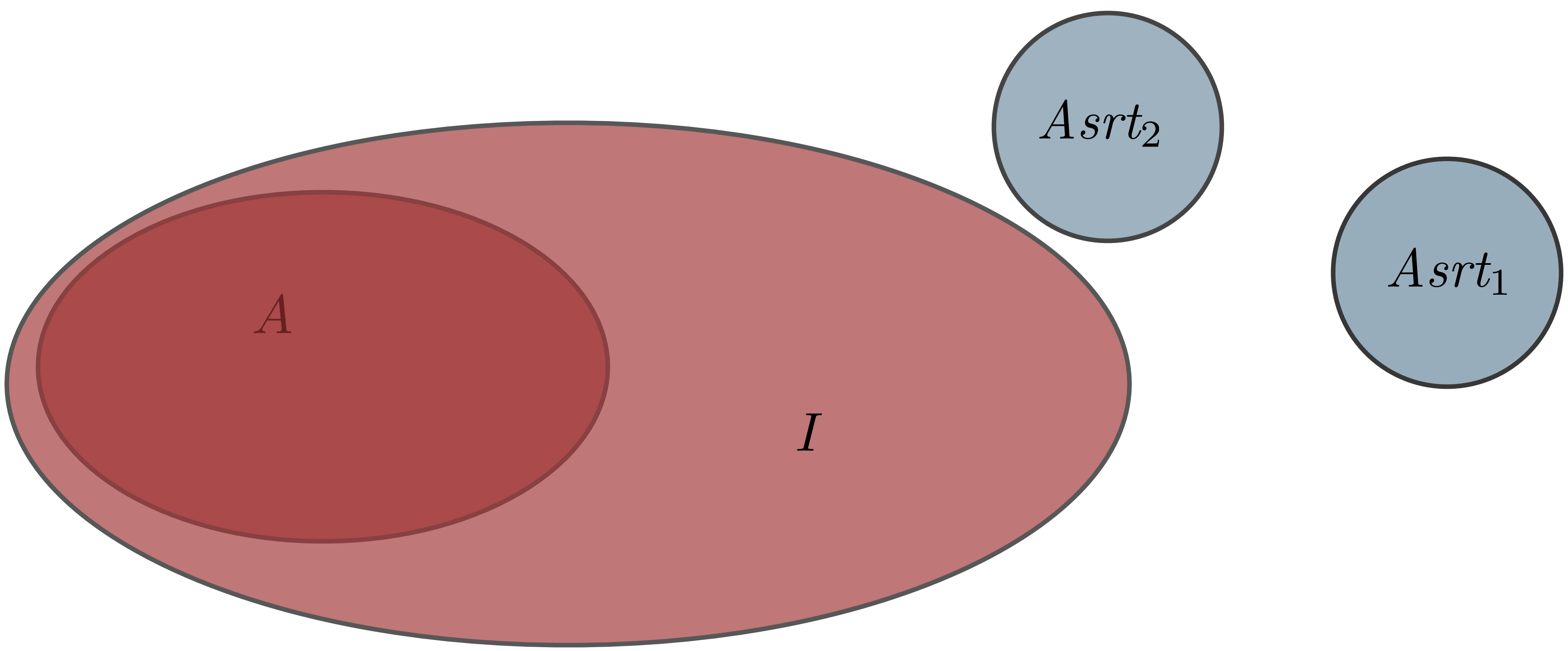

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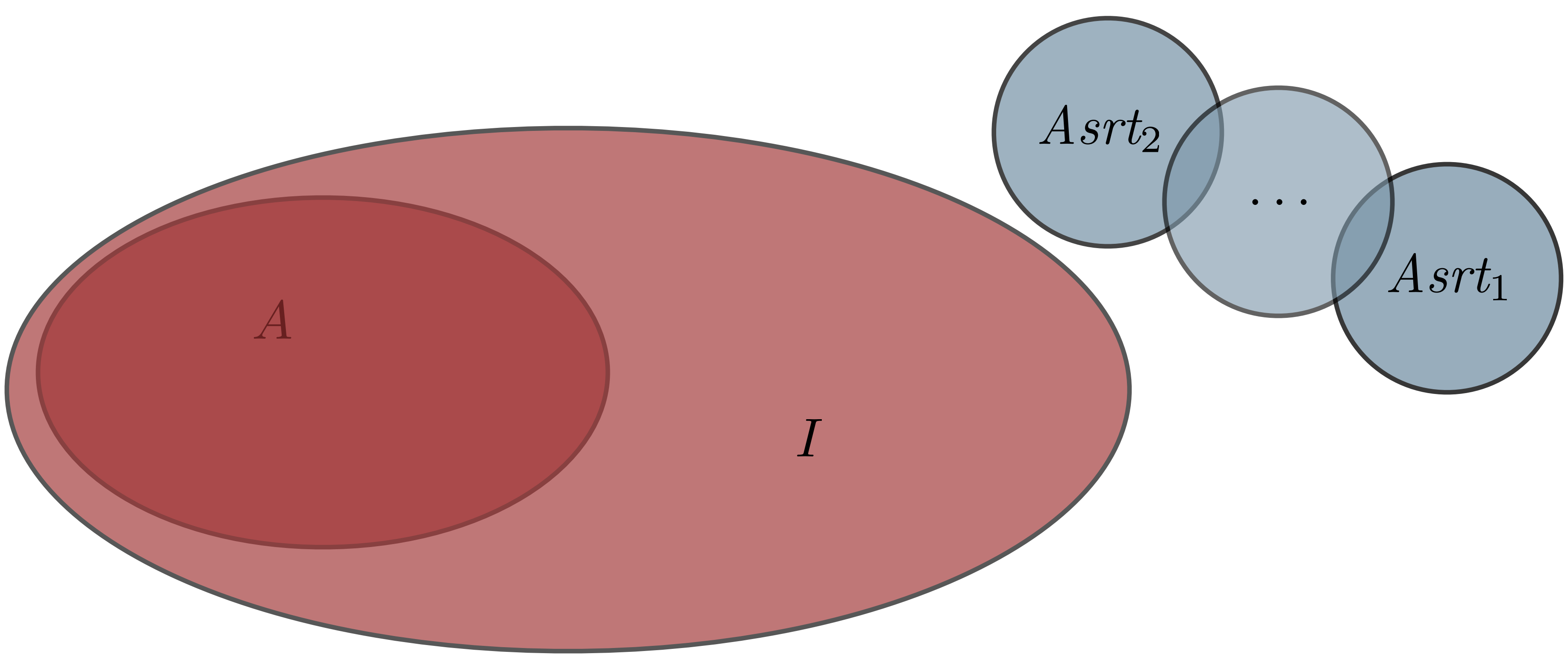

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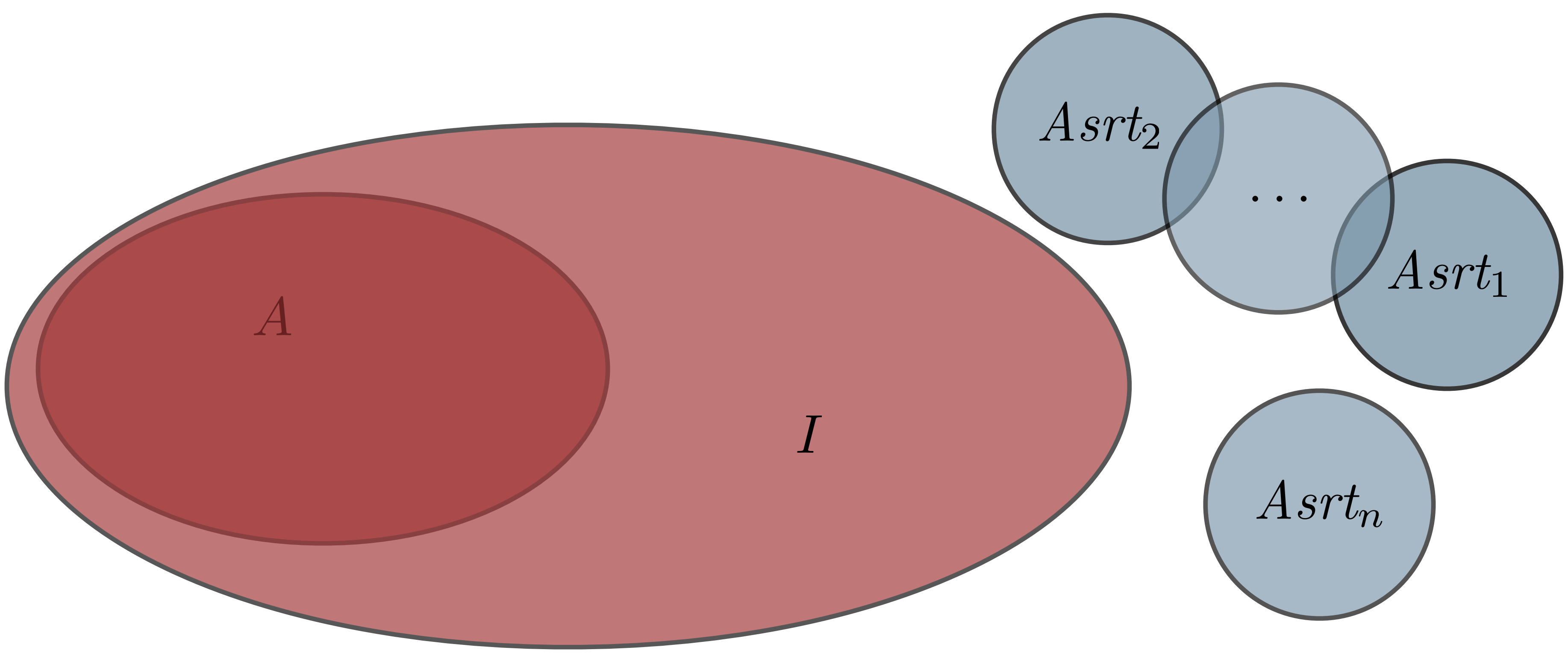

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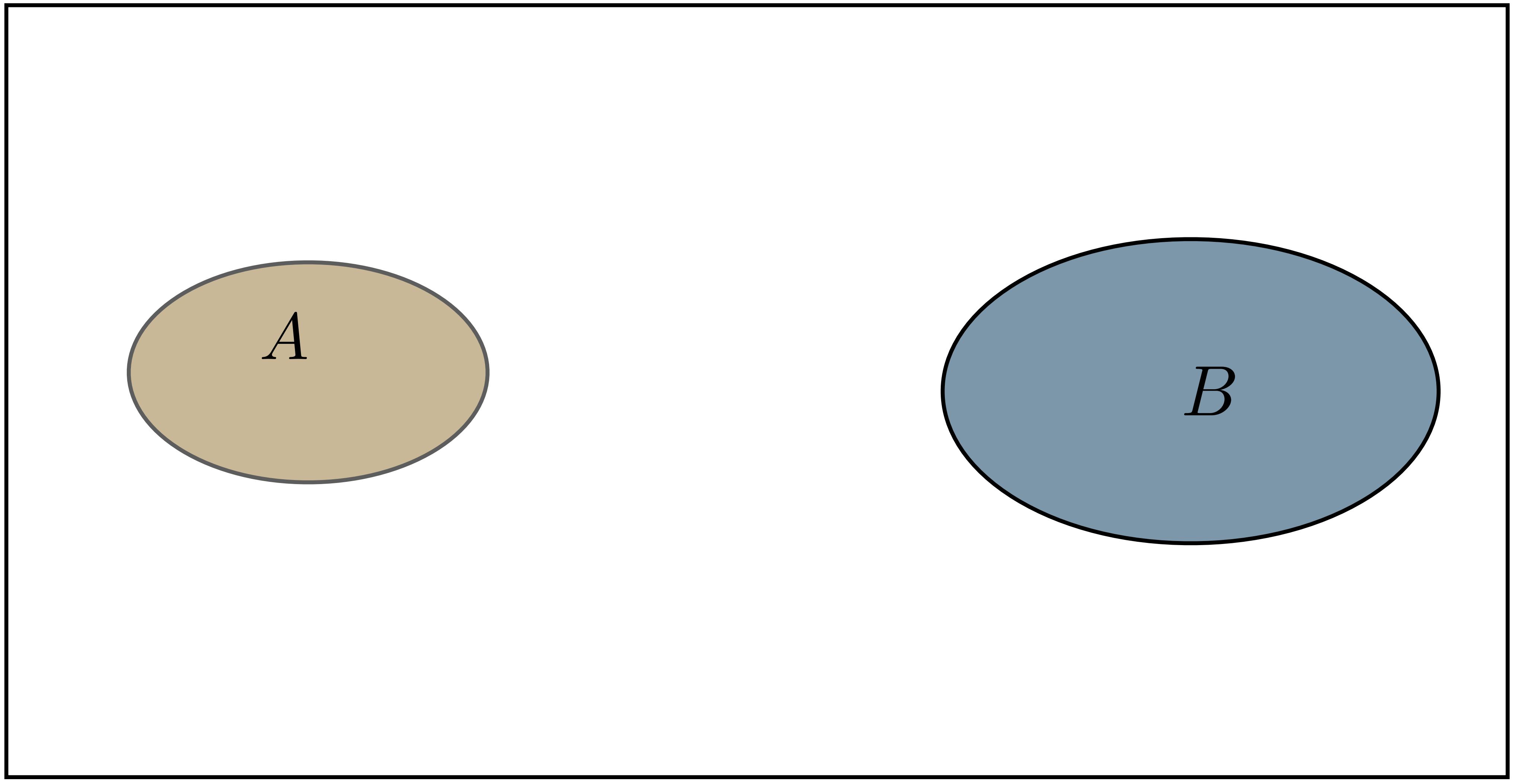

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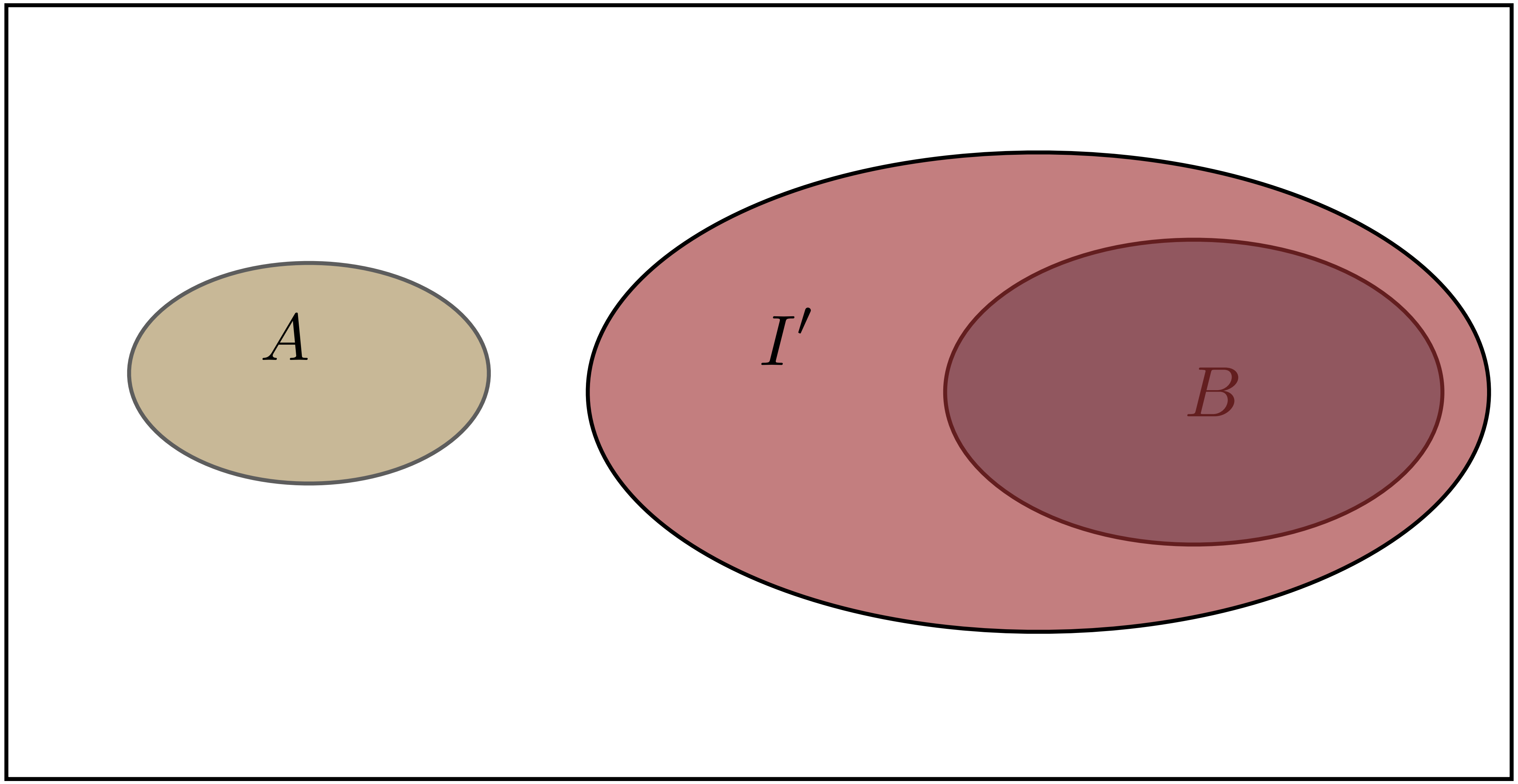

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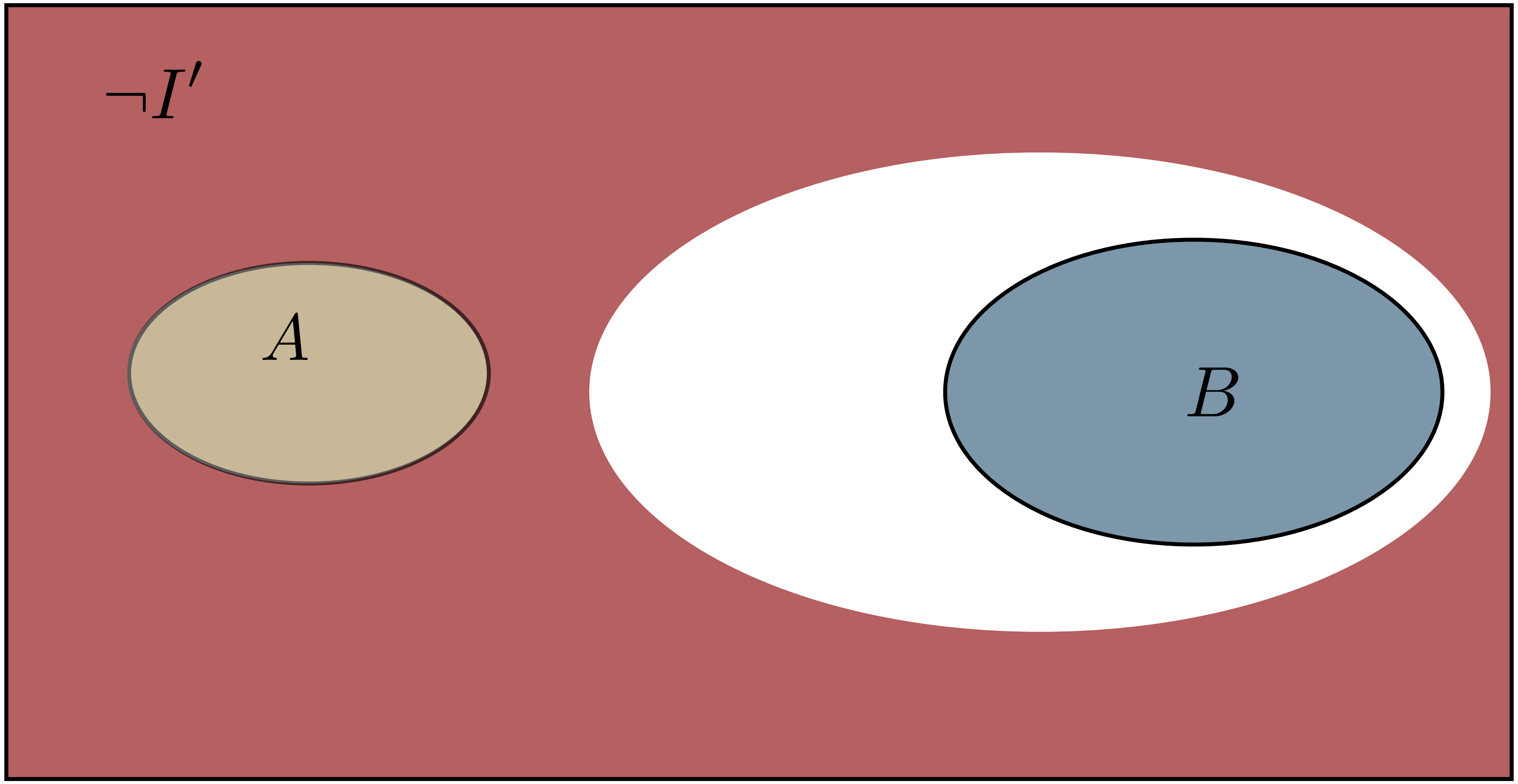


# What is LRA

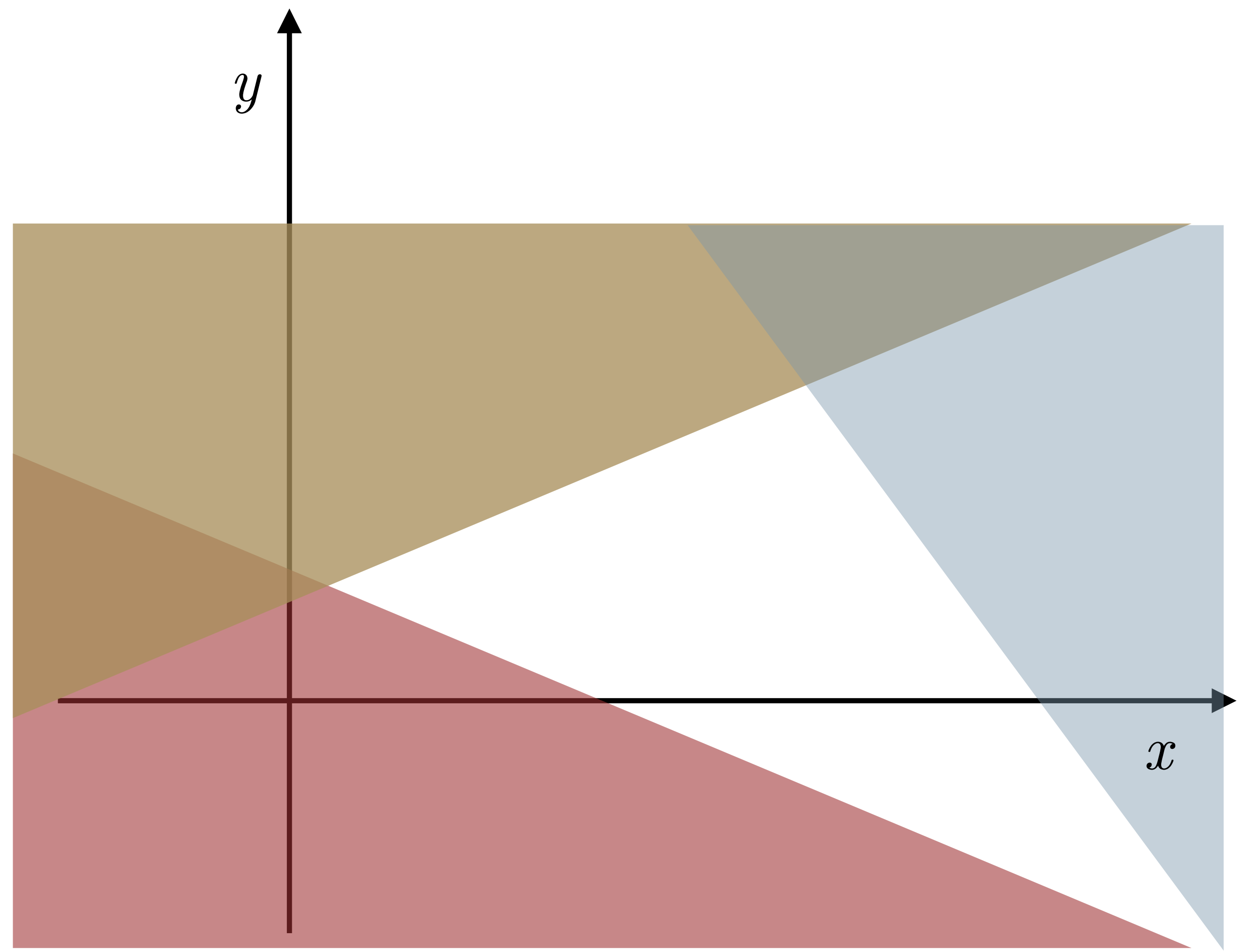
Given a set of linear inequalities over real-valued variables,

determine if there are values for the variables that satisfy all the inequalities

$$0 \leq 0.5x + 3y - 2z$$

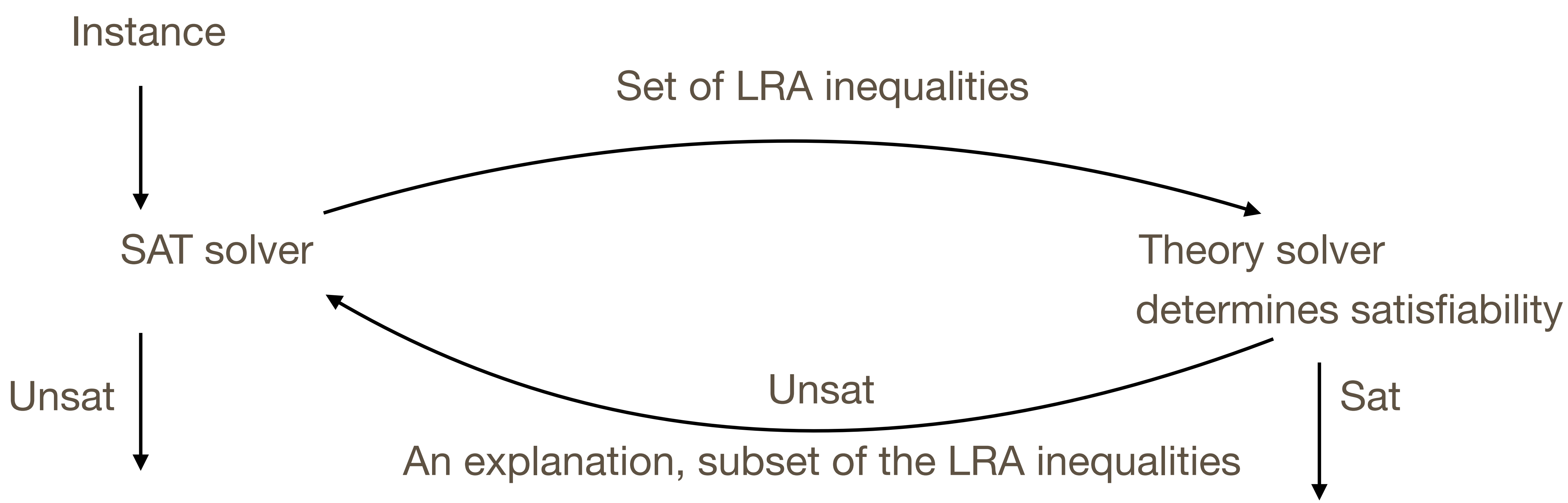
$$5 > y$$

$$\cancel{x^2 + 2xy + y^2 > 1}$$




In 2 dimensions: determine whether half planes have a non-empty intersection

# Solving LRA in SMT




The theory solver for LRA is based on the Simplex algorithm

# Simplex in SMT

A pre-processing step:

- All inequalities are written so that left side is a constant and right side a linear expression

We end up with two types of entities:

- Bounds on variables
- Bounds on sums of the variables

The idea is to repeatedly adjust variable values to satisfy bounds on the sums, and change the role of the variables and the sums.

$$0 \leq x + 6y - 4z$$

$$0.3 \geq x + 2y$$

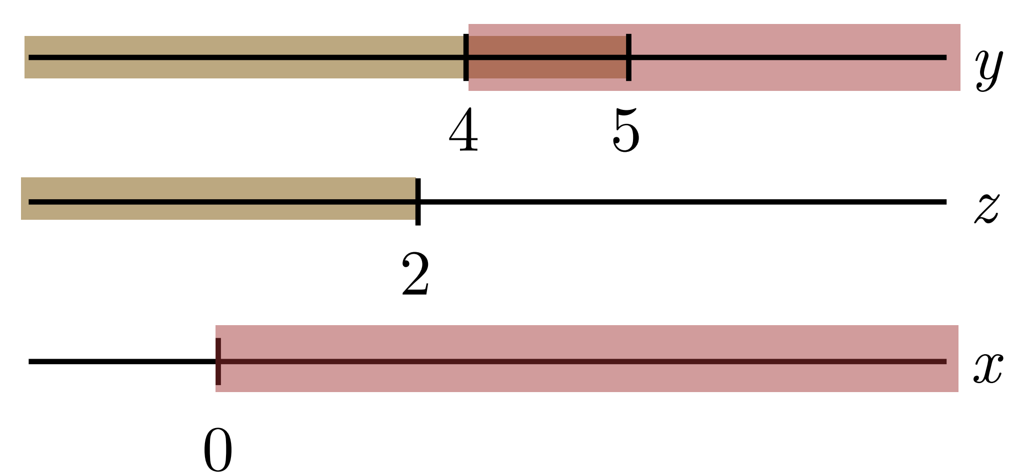
$$1 < x - 2y + 3z$$

$$5 > y$$

$$4 < y$$

$$2 > z$$

$$0 \geq x$$



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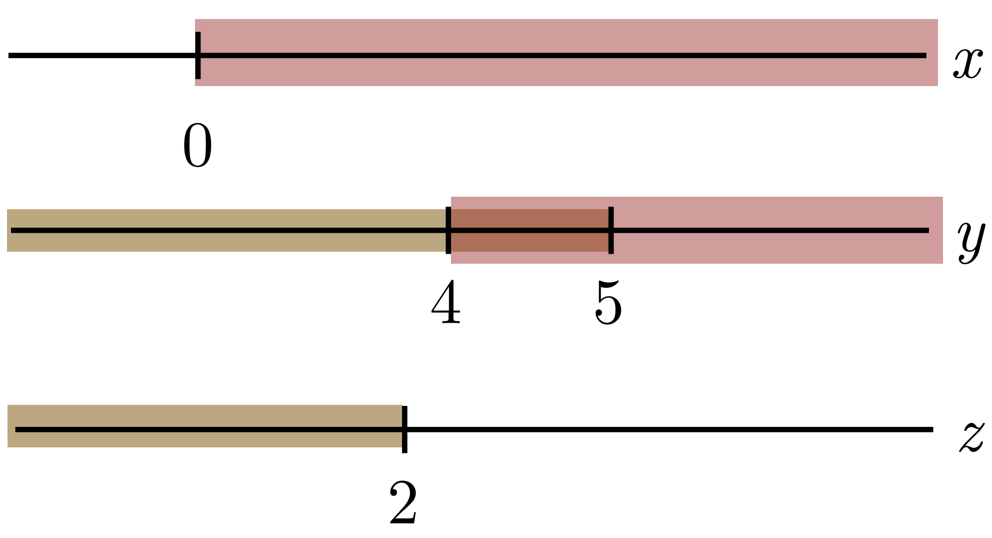

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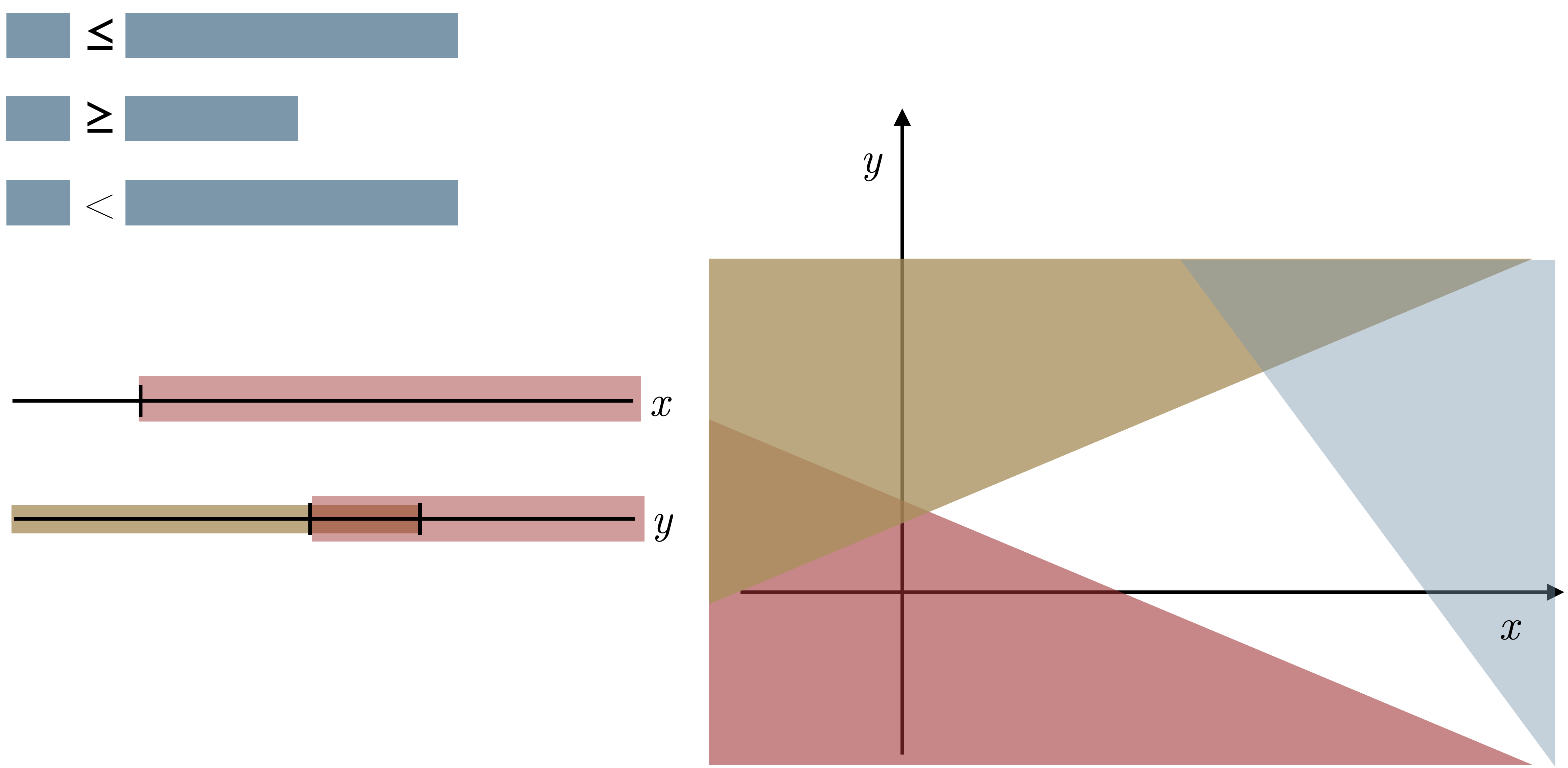

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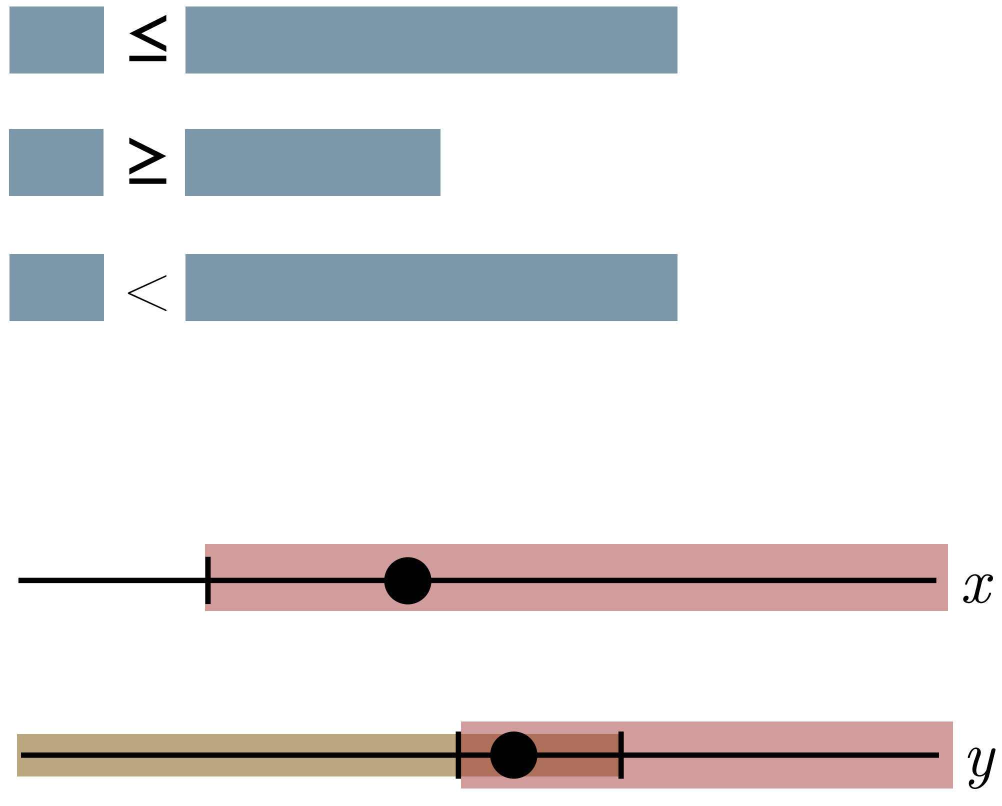


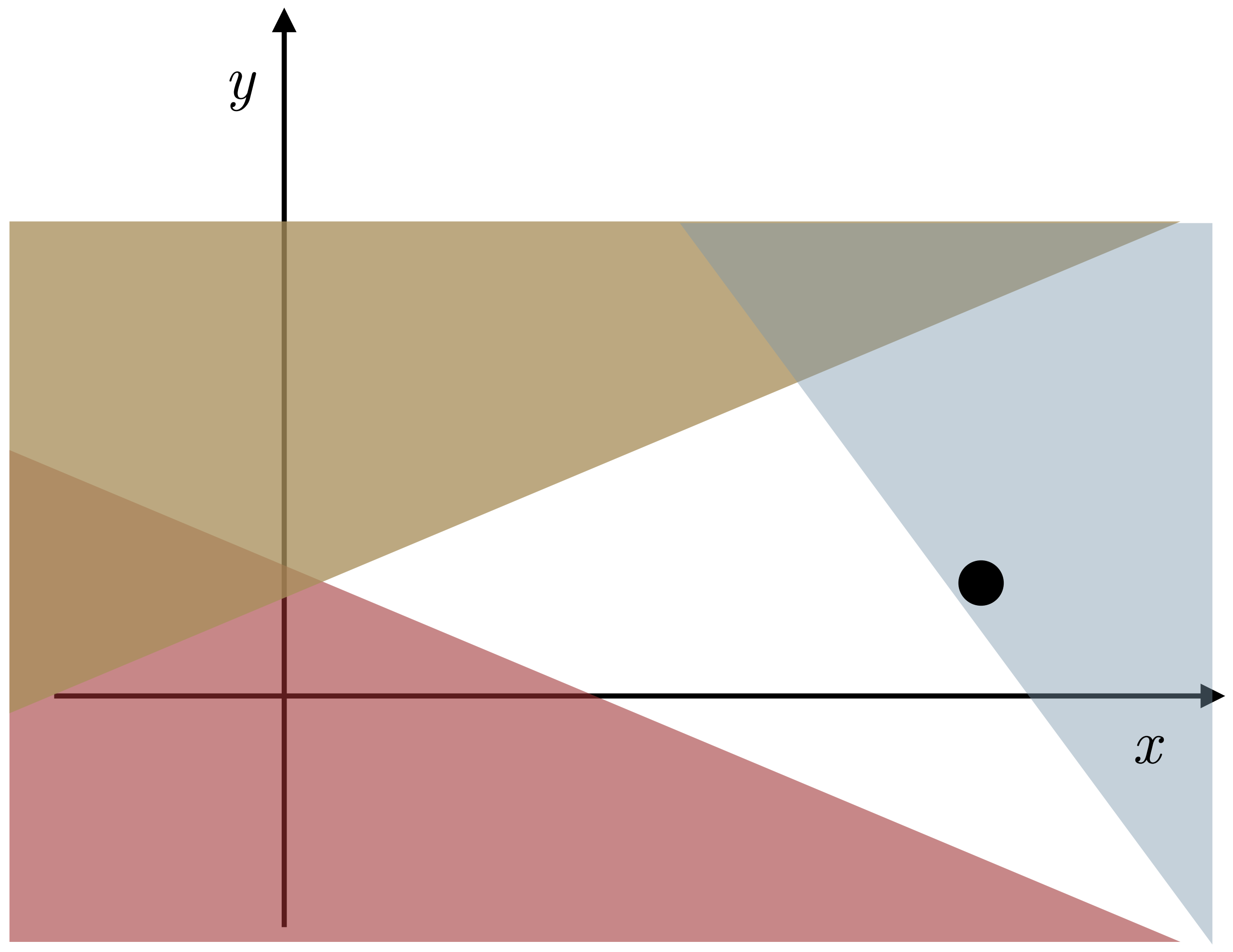

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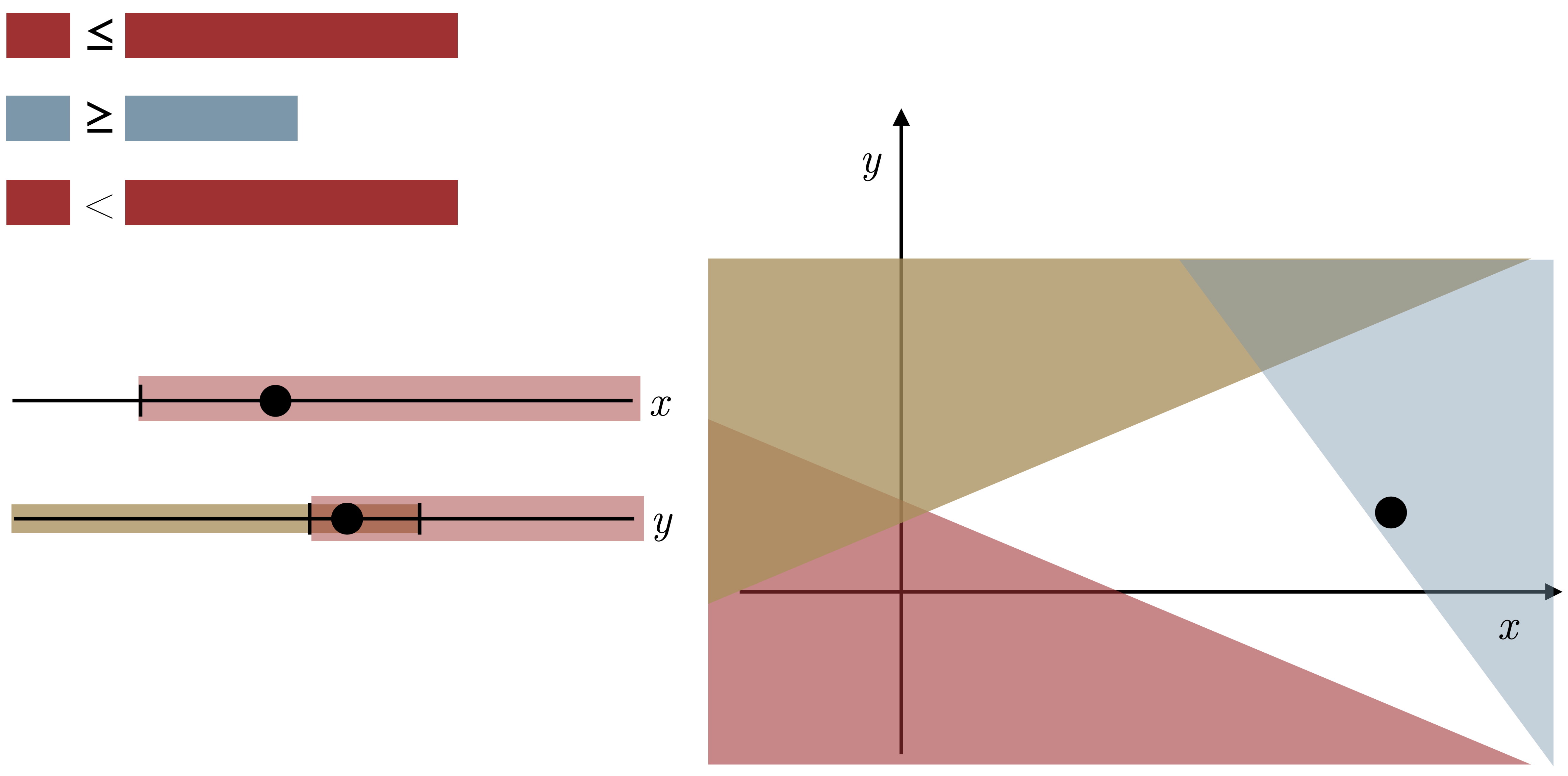

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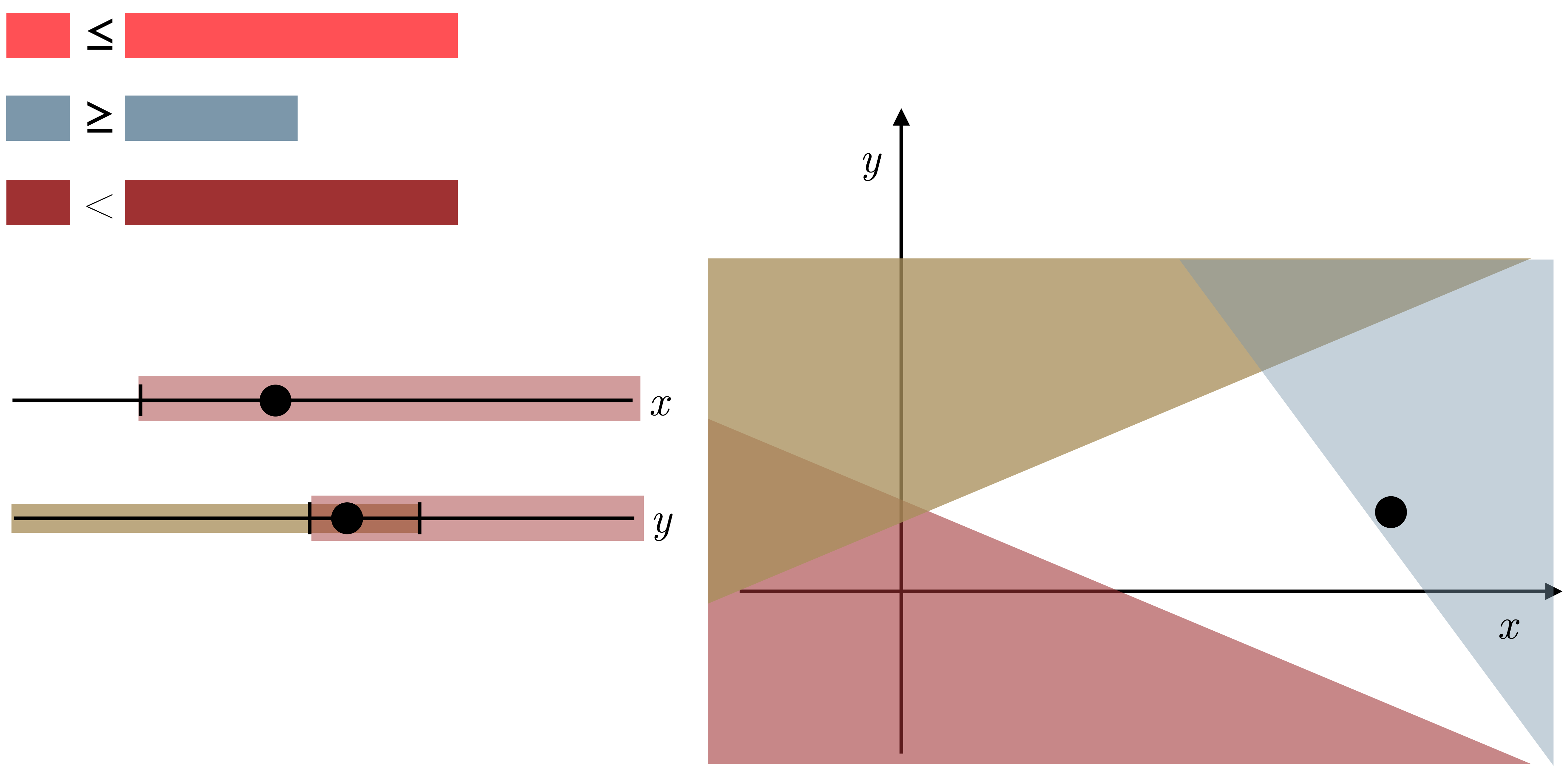

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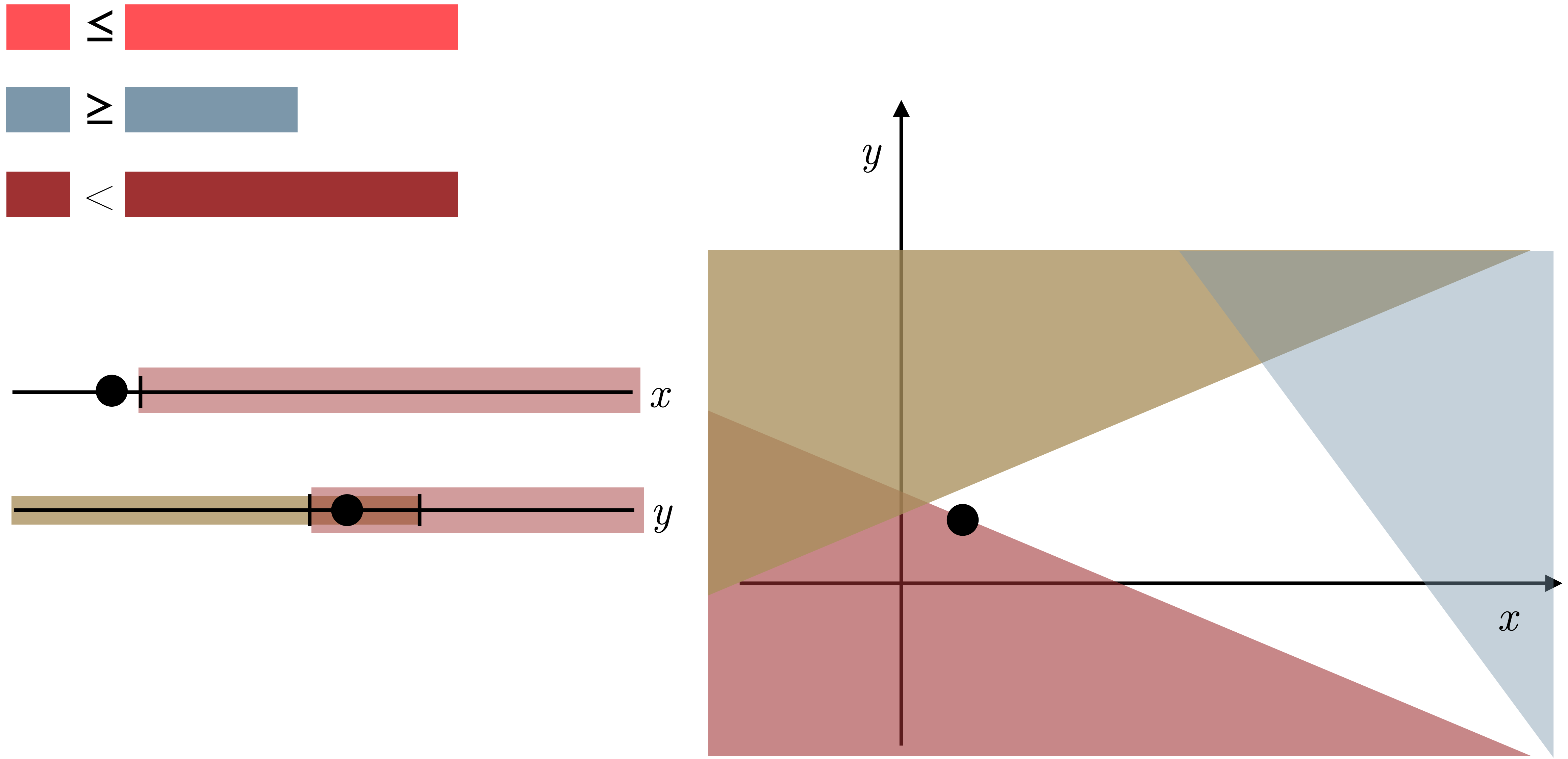

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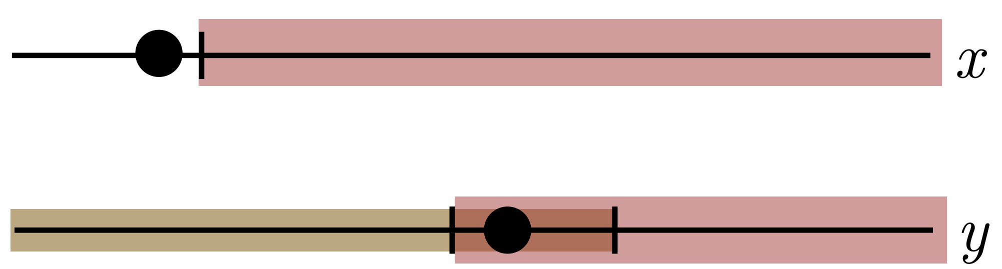


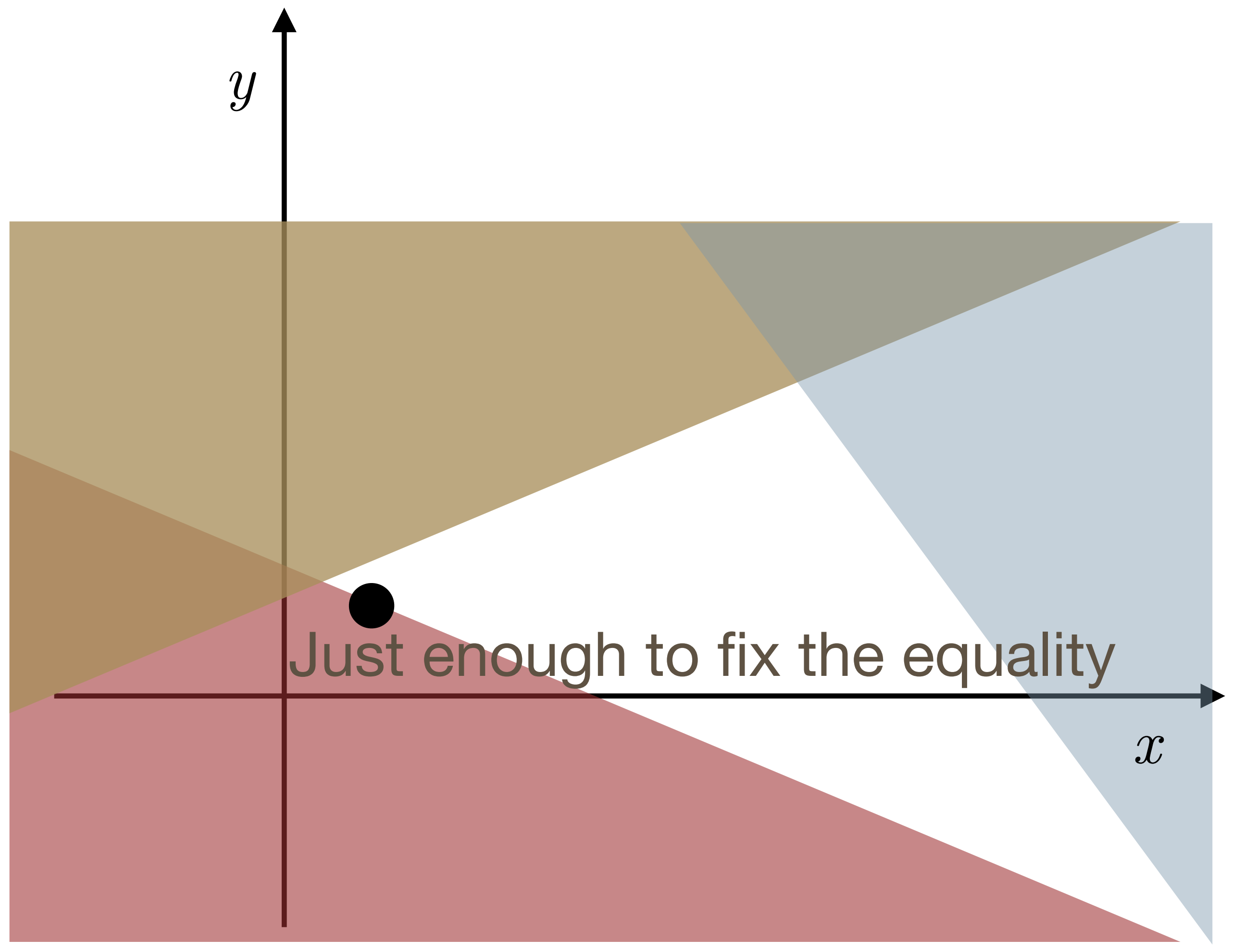

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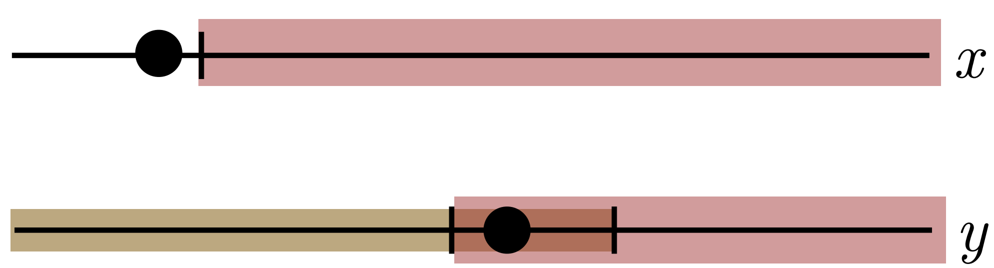


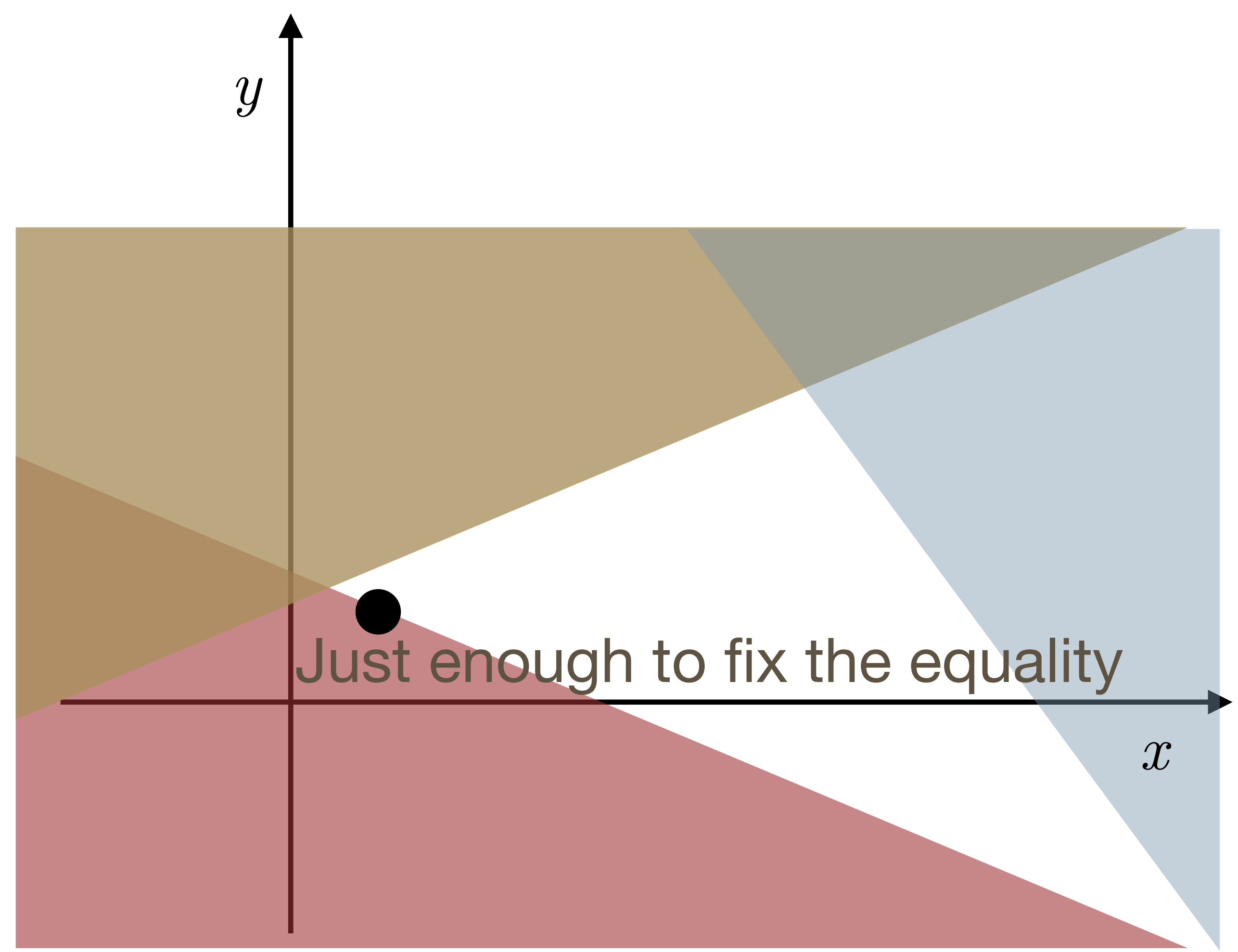

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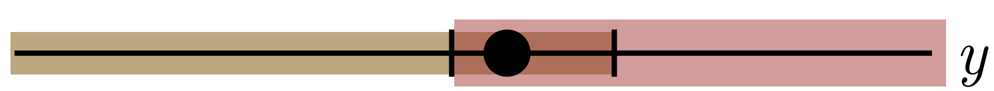


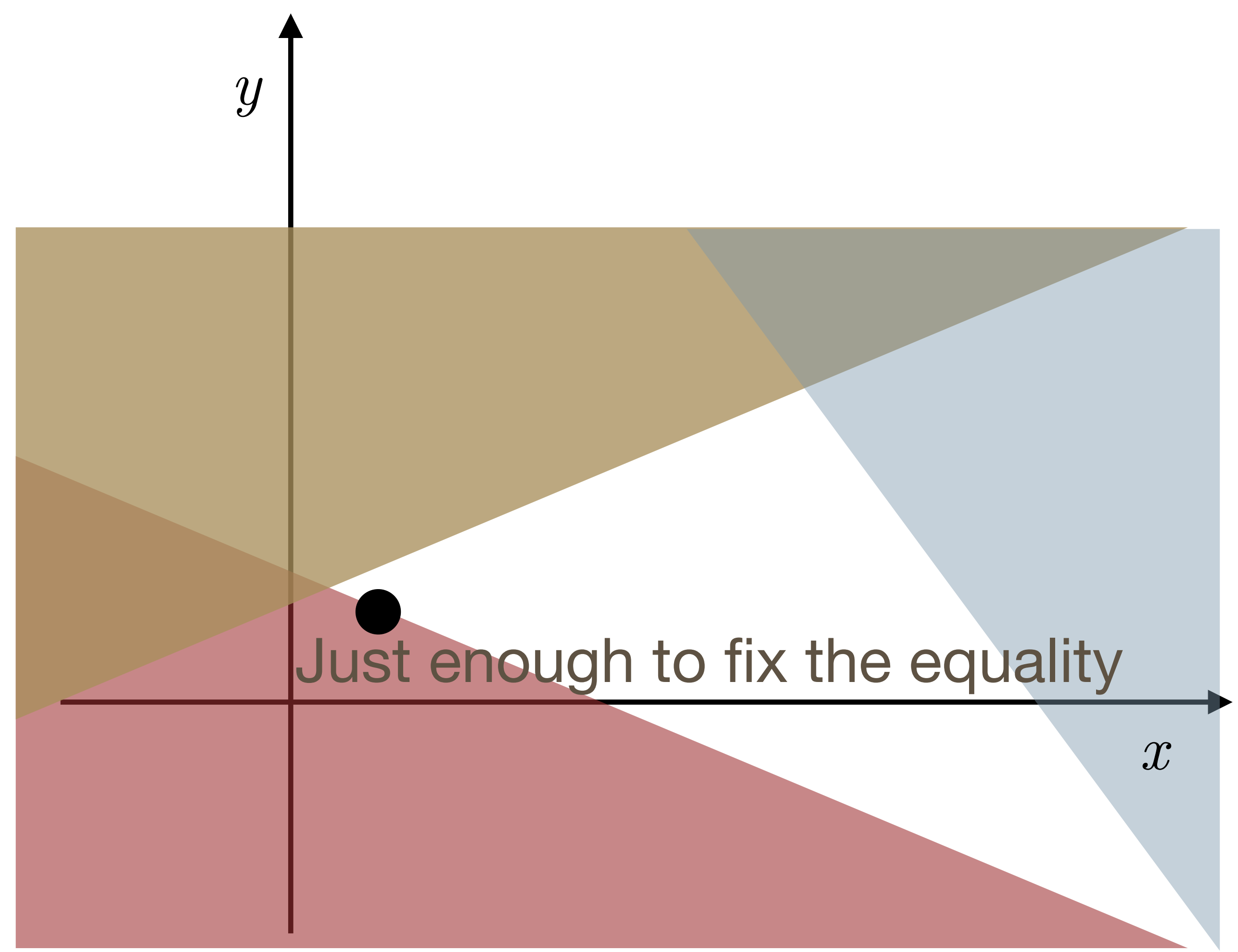

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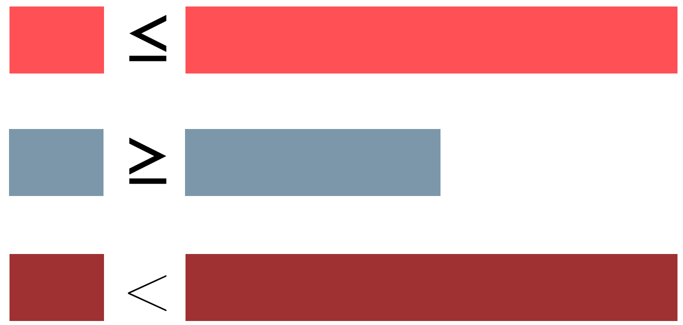


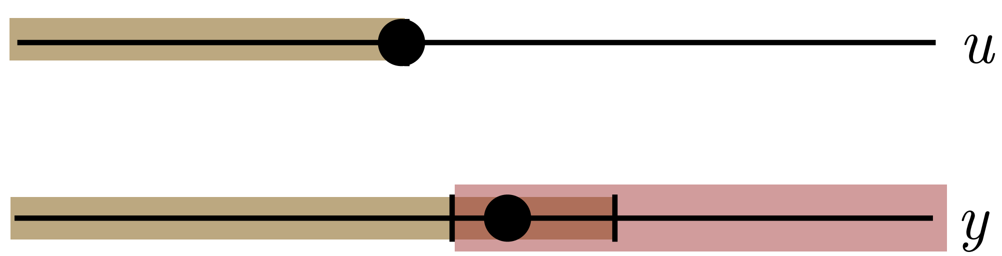


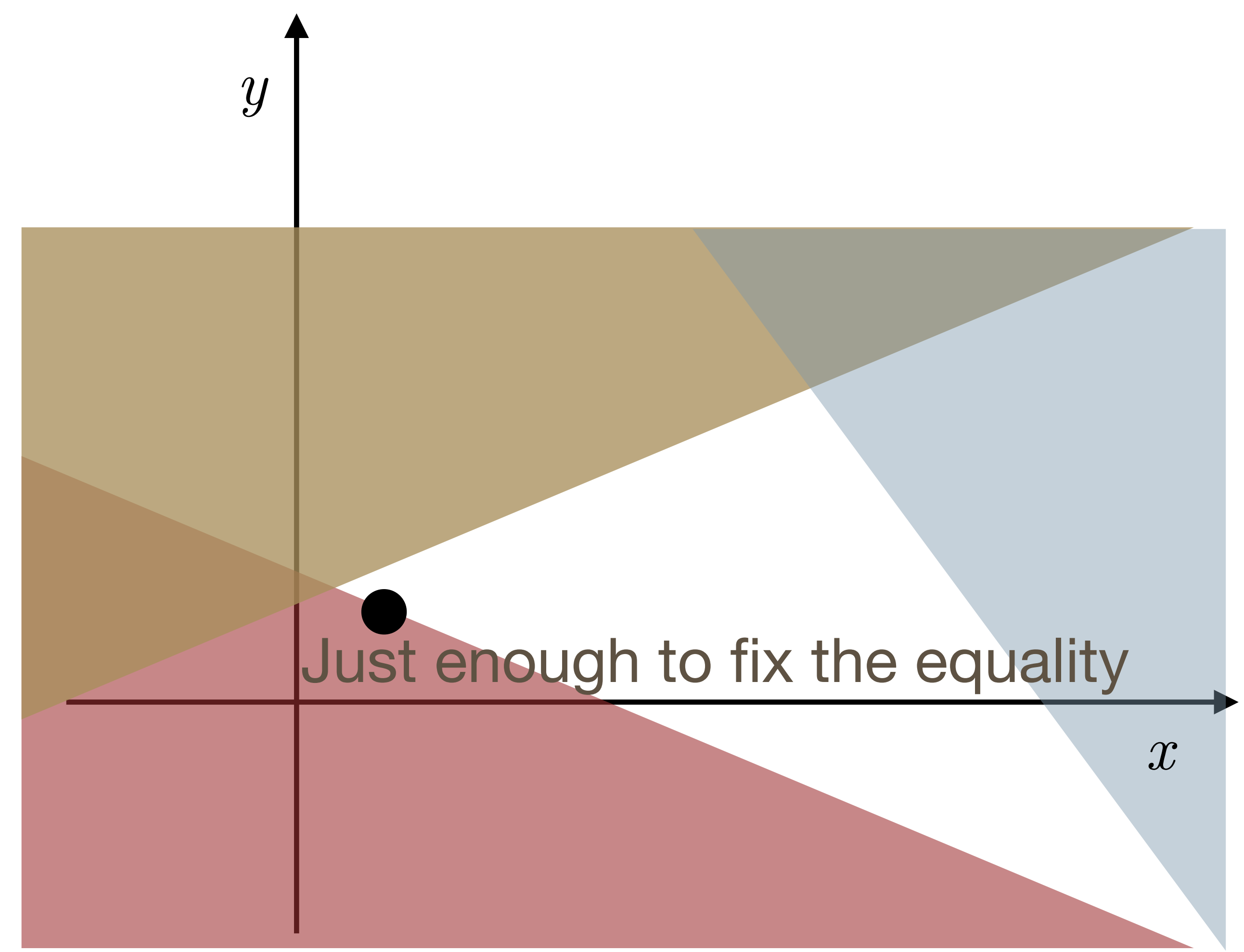

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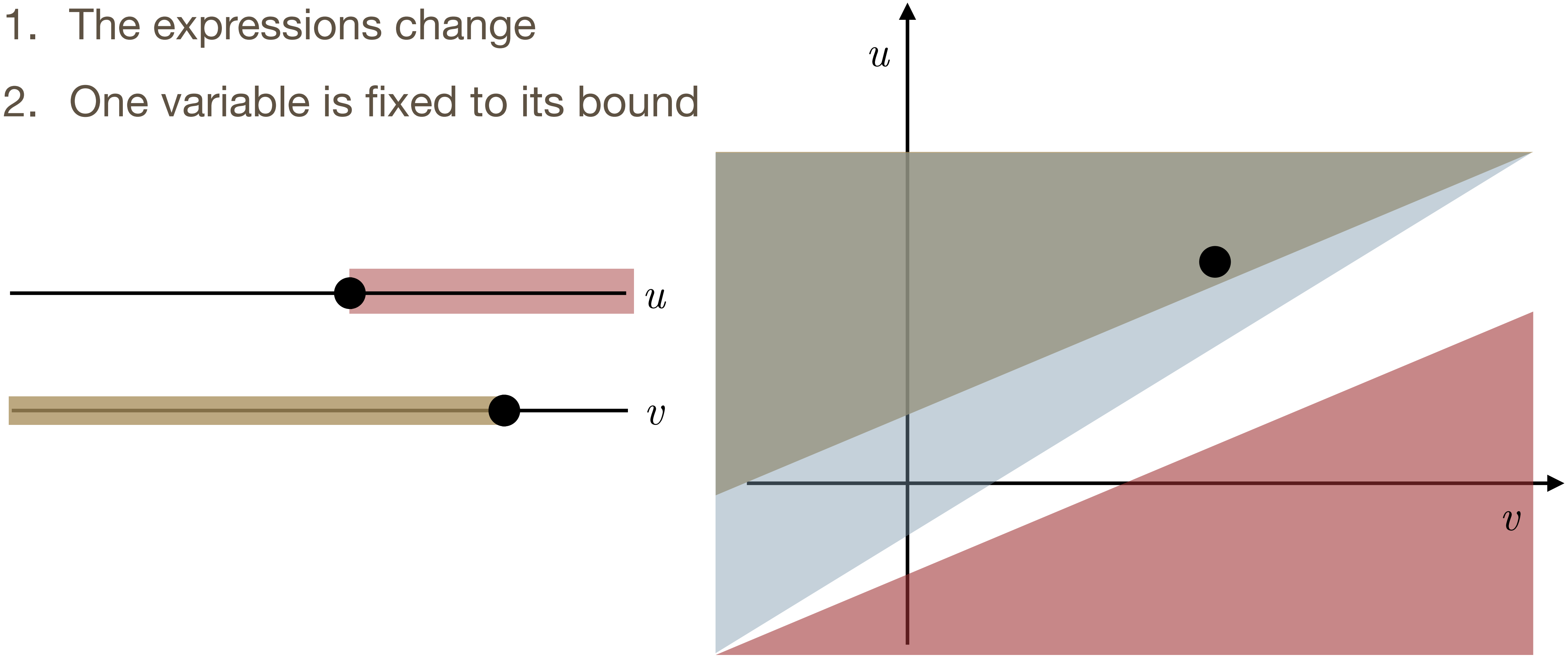

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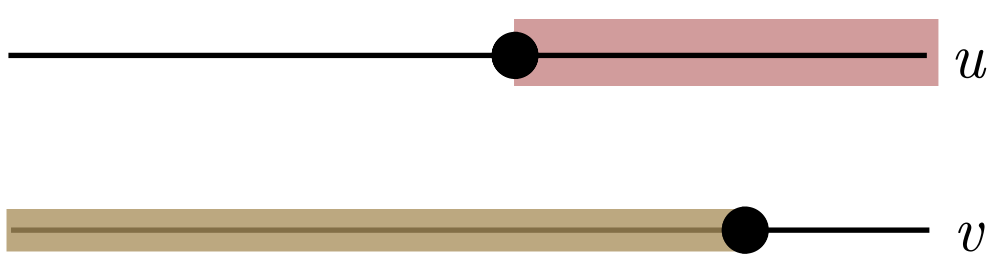


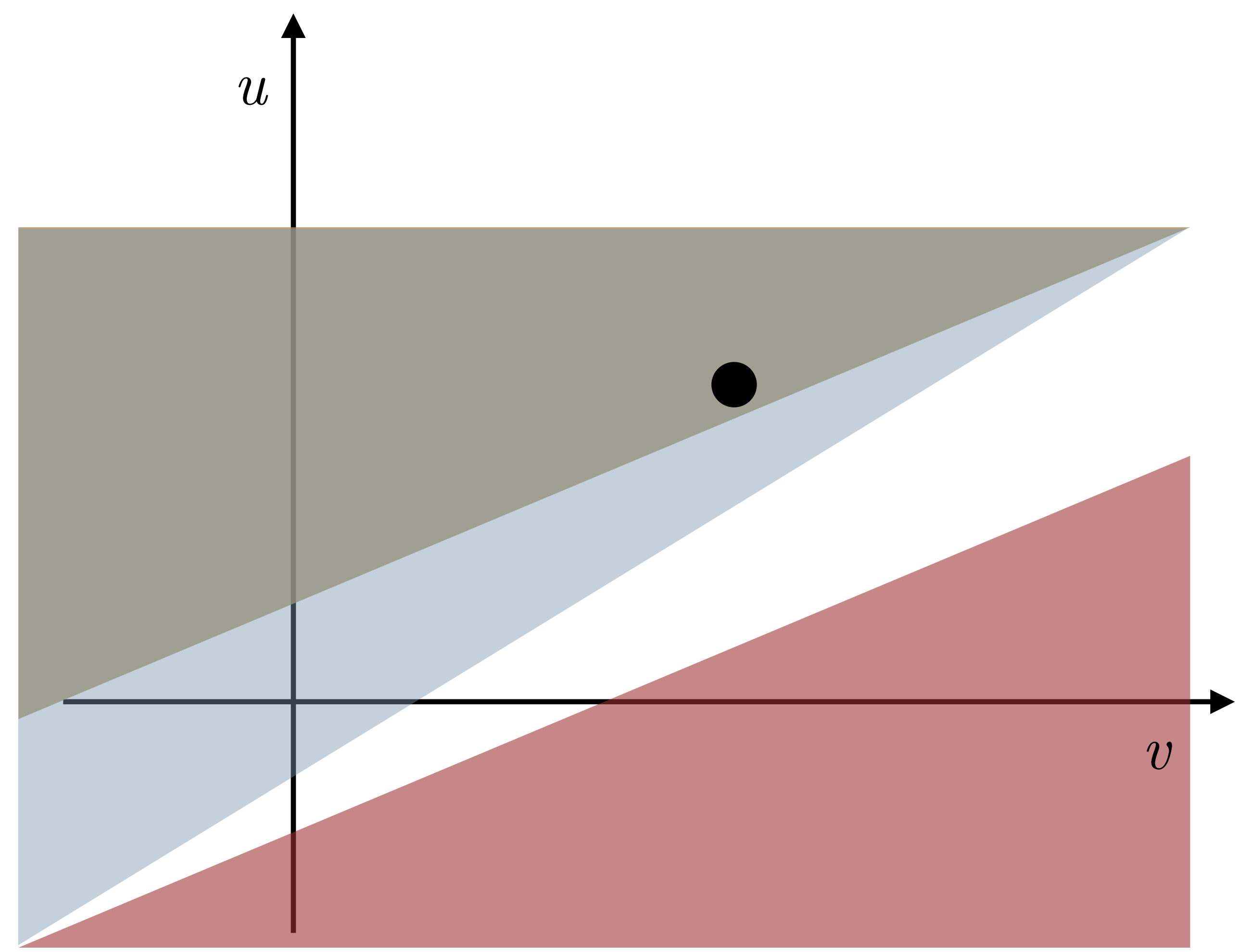


If an expression bound cannot be satisfied since the variables are at their bounds, the problem is unsatisfiable

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A conflict is the unsatisfied expression and the set of expressions currently bounding its variables.

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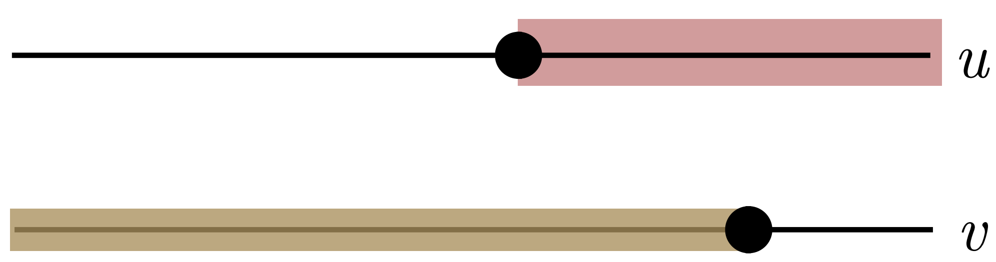


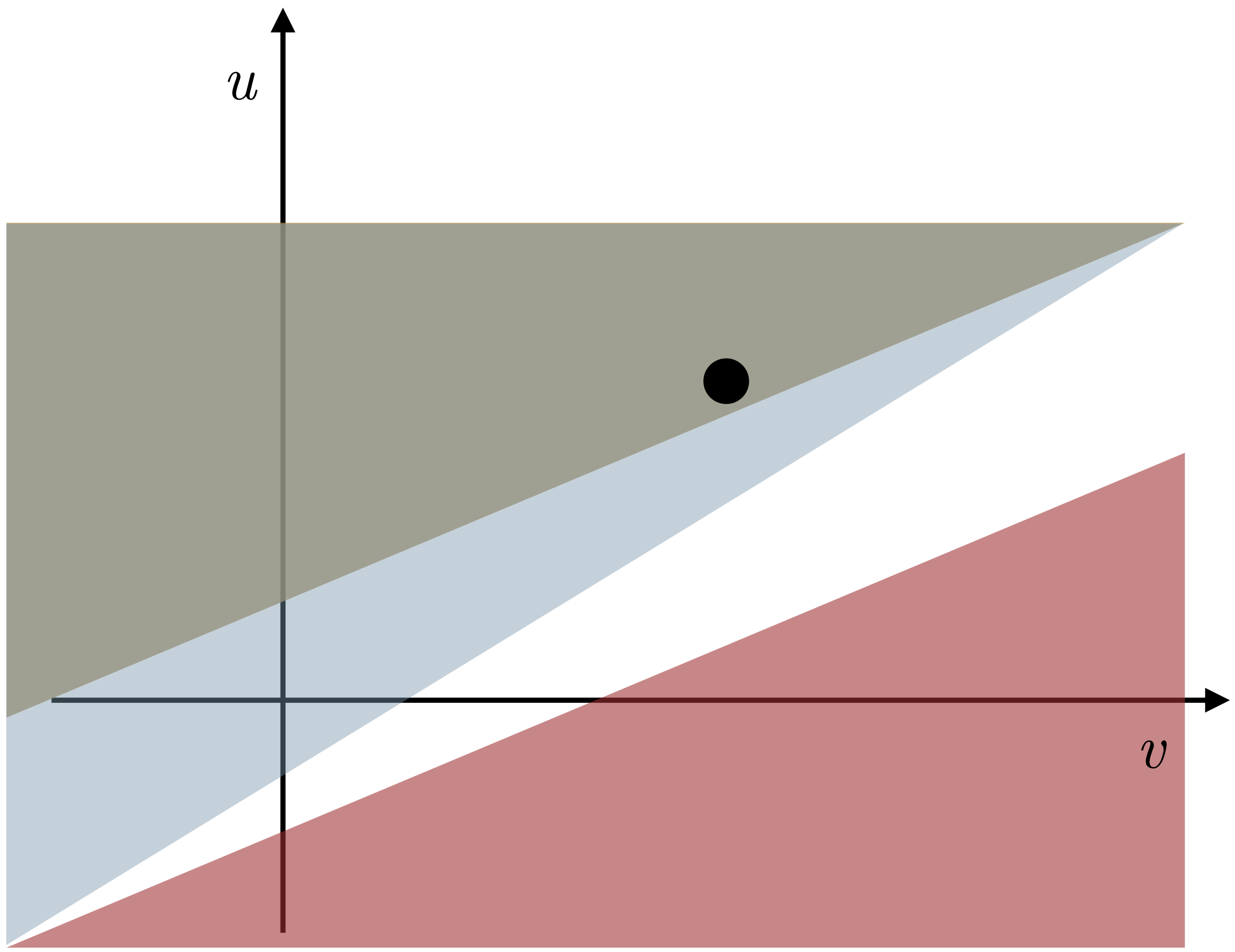


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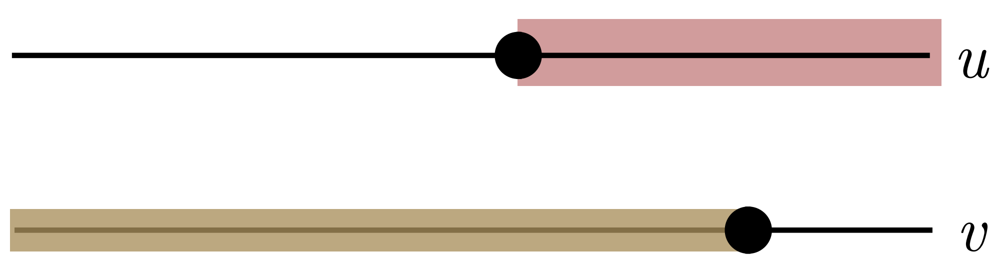


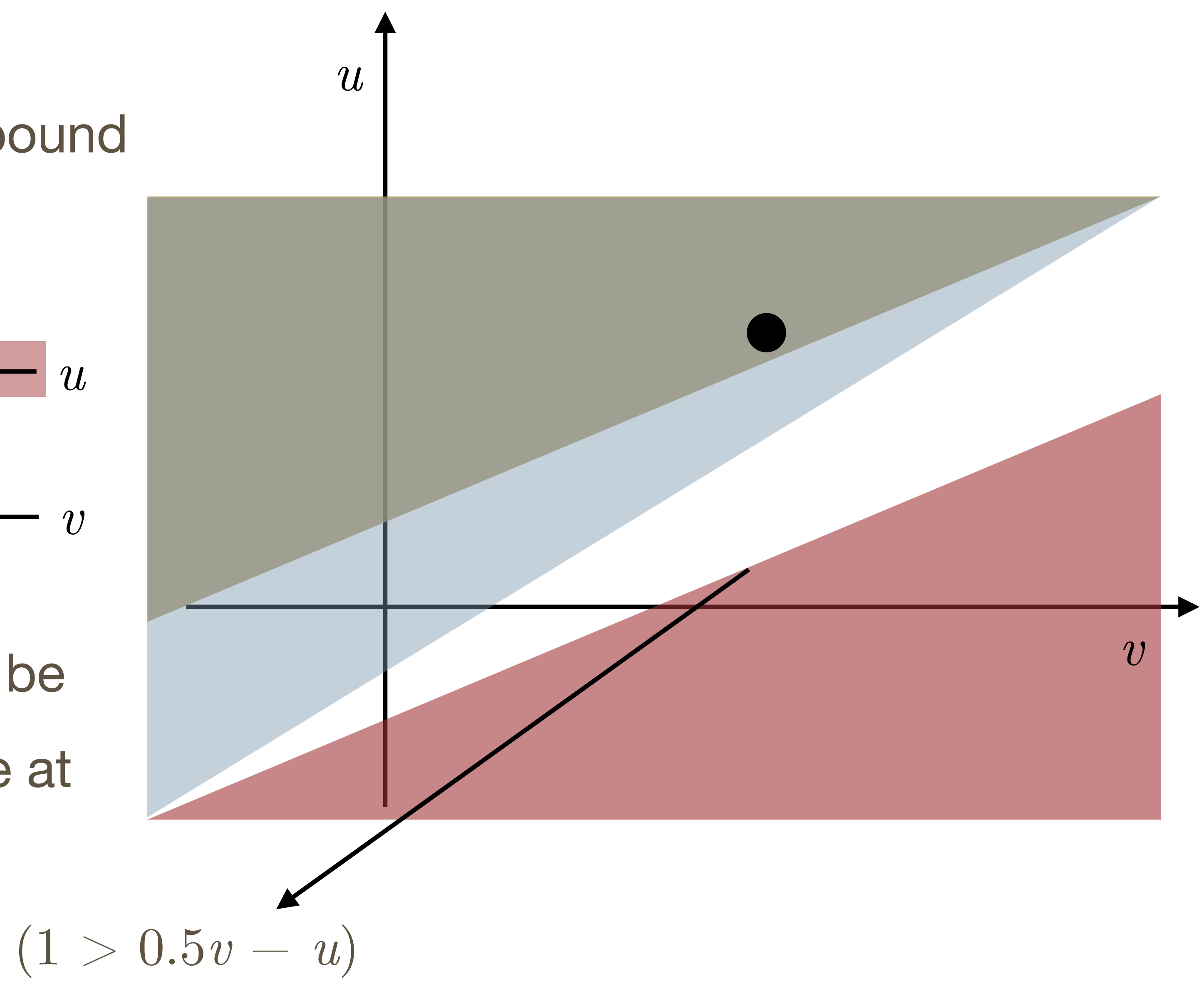


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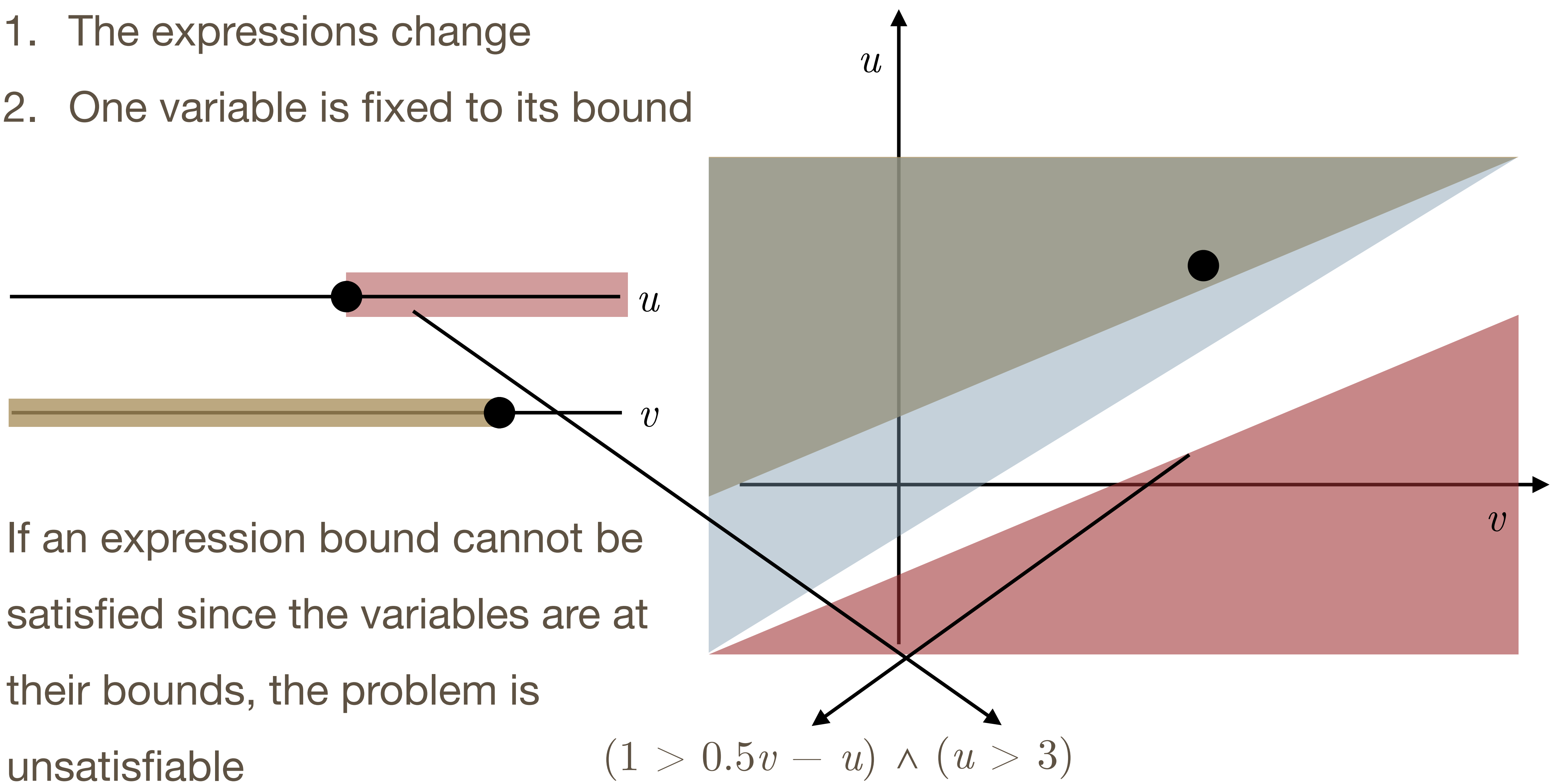


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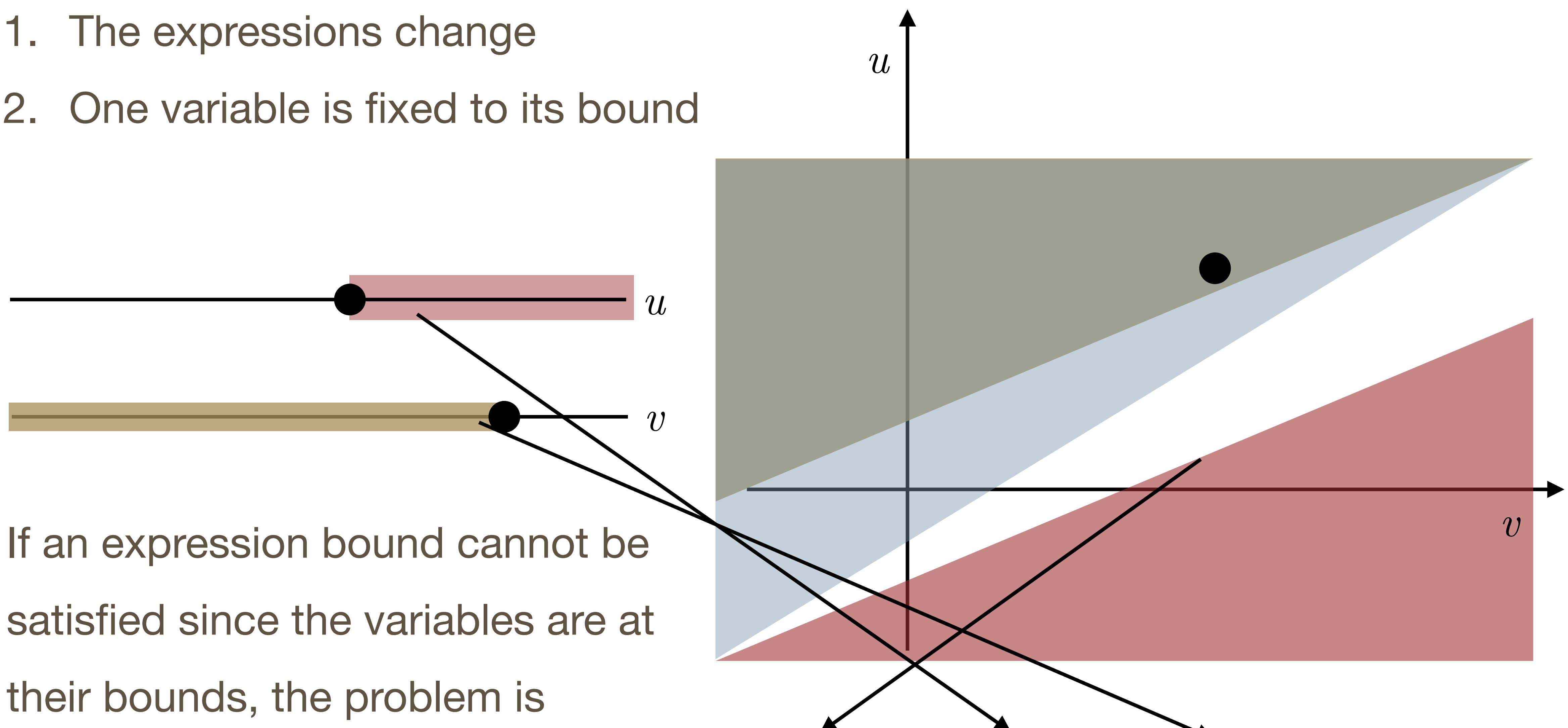


$(1 > 0.5v - u) \wedge (u > 3) \wedge (v < 3)$

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# LRA Interpolation

Assume that the expression

bound that could not be satisfied

was $1 > 0.5v - u$

and the bounds for the variables

$u, \ v$ were

$u > 3$

$v < 3$

Assume that $(u > 3) \in B$ and

$(v < 3) \in A$, $(1 > 0.5v - u) \in A$.

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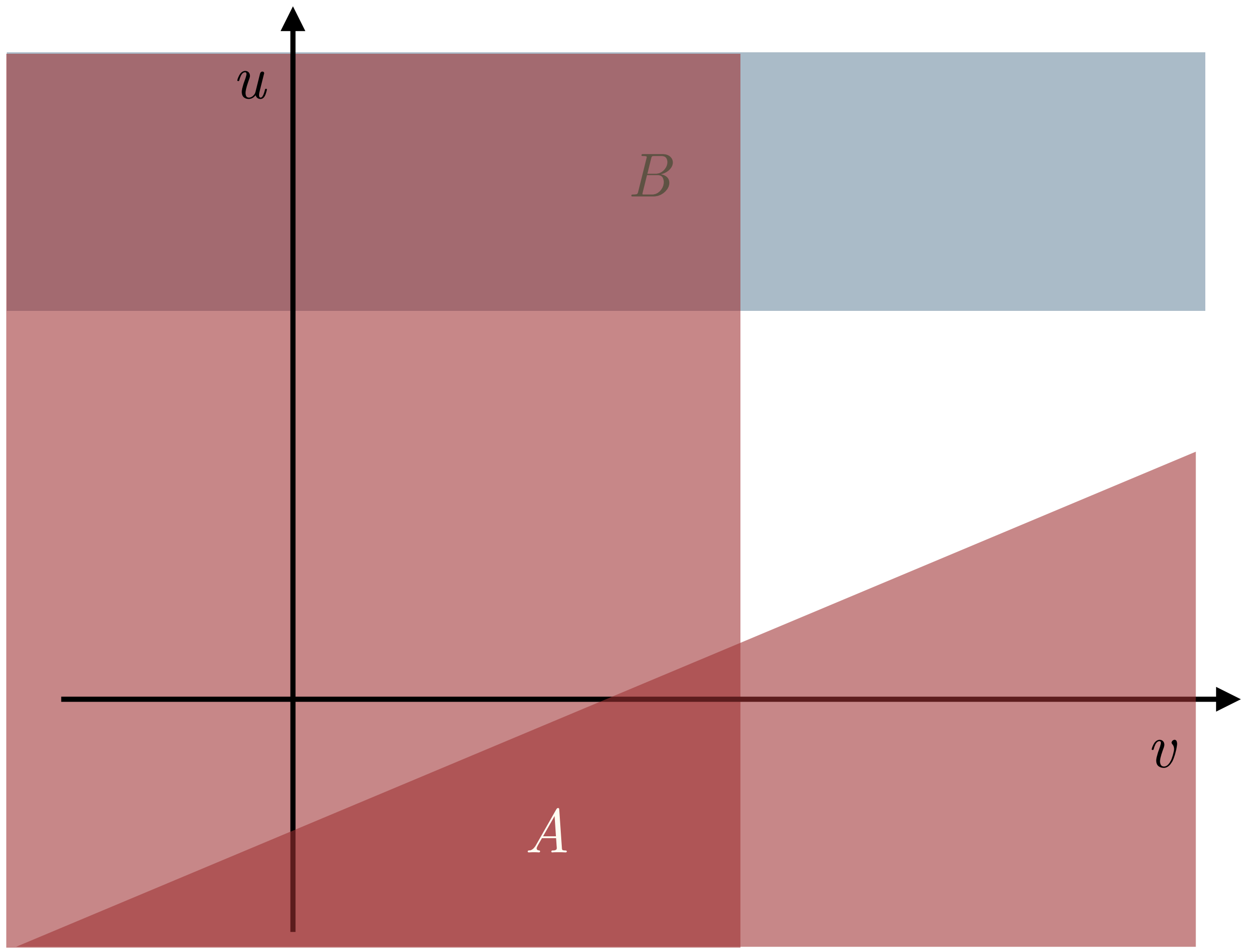

# LRA Interpolation

The interpolant for $A$ is obtained by summing to the expression the bounds in $A$ multiplied by their factors in the expression:

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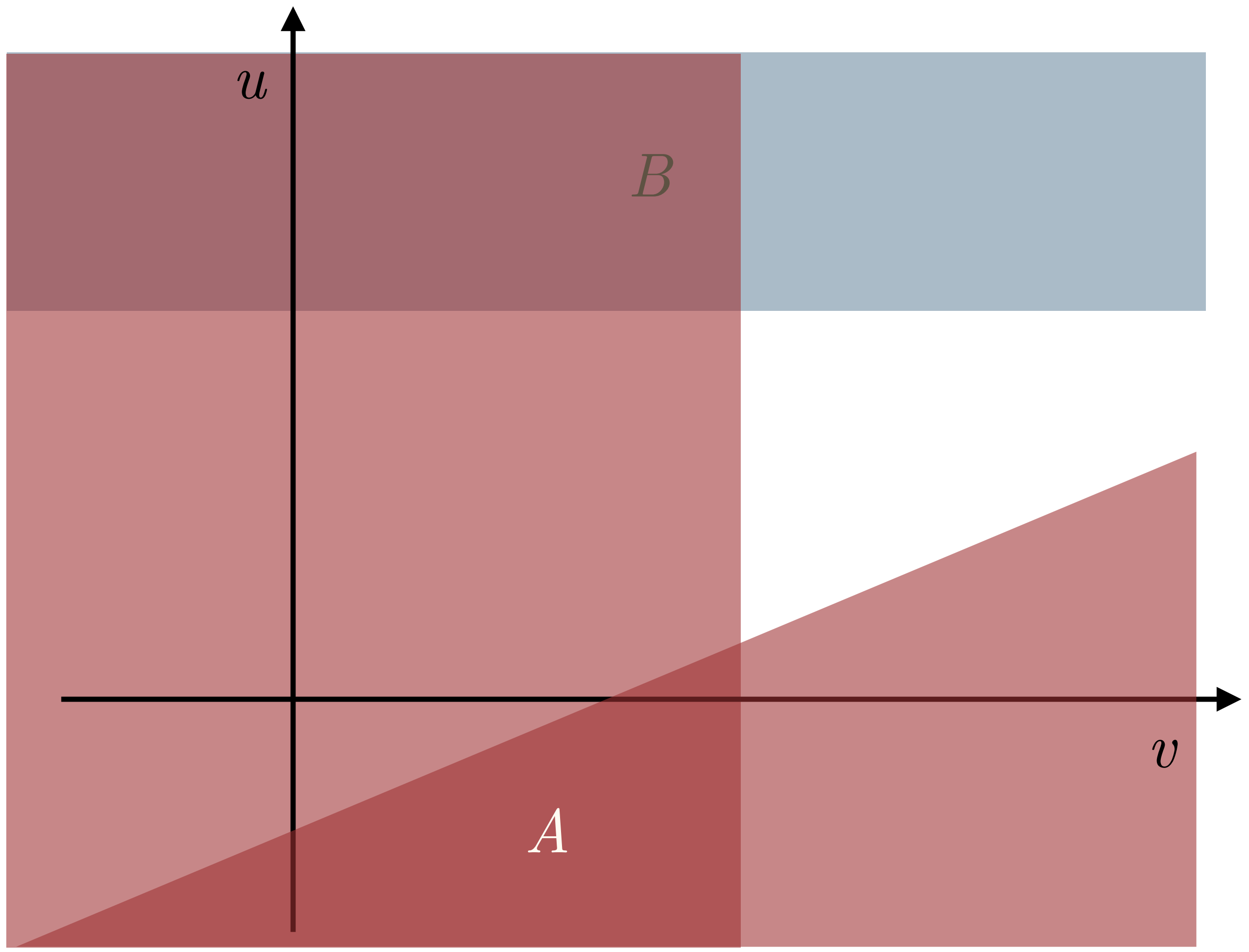

# LRA Interpolation

The interpolant for $A$ is obtained by summing to the expression the bounds in $A$ multiplied by their factors in the expression:

$(0.5v - u - 1)$

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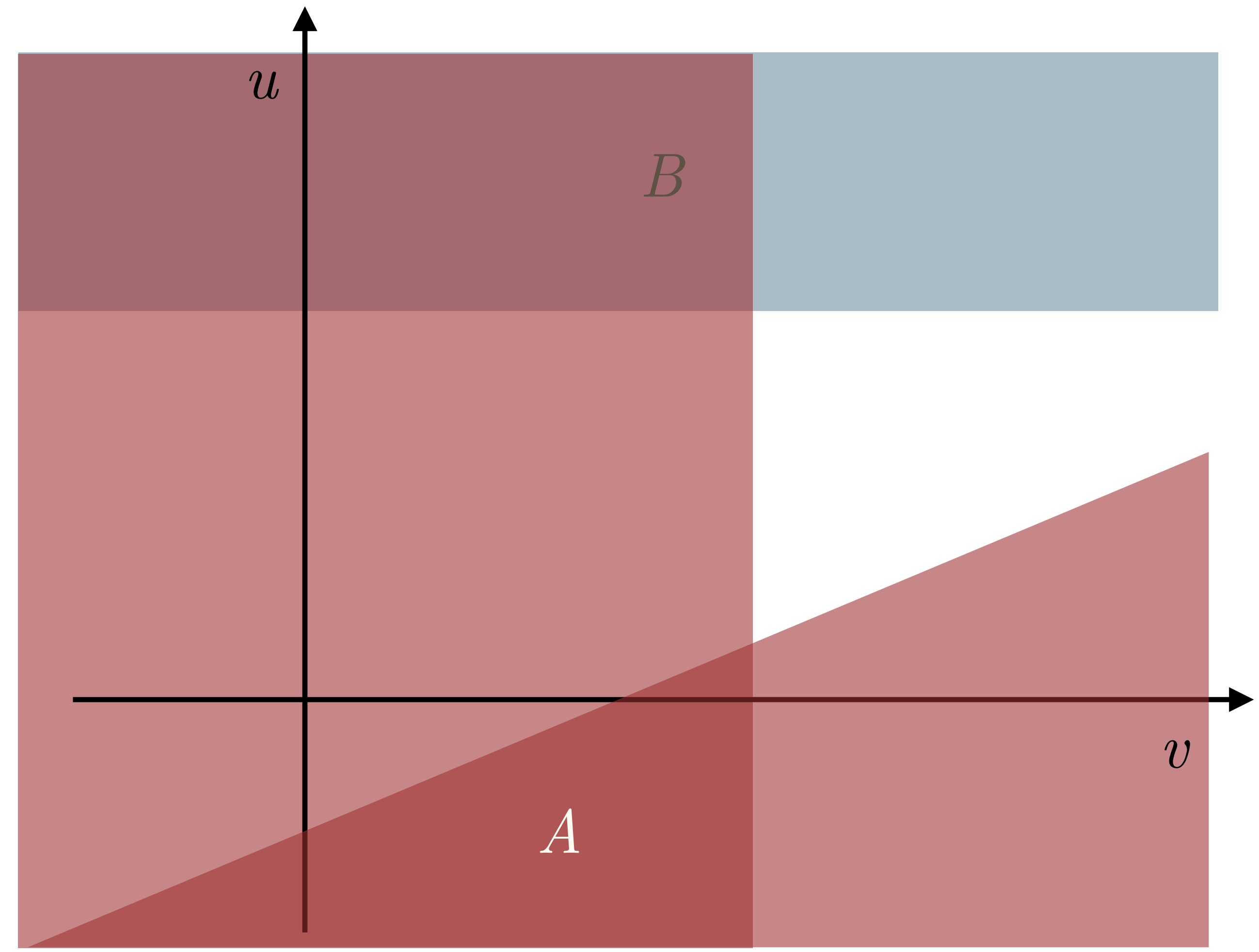

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$$(0.5v - u - 1) + 0.5(3 - v)$$

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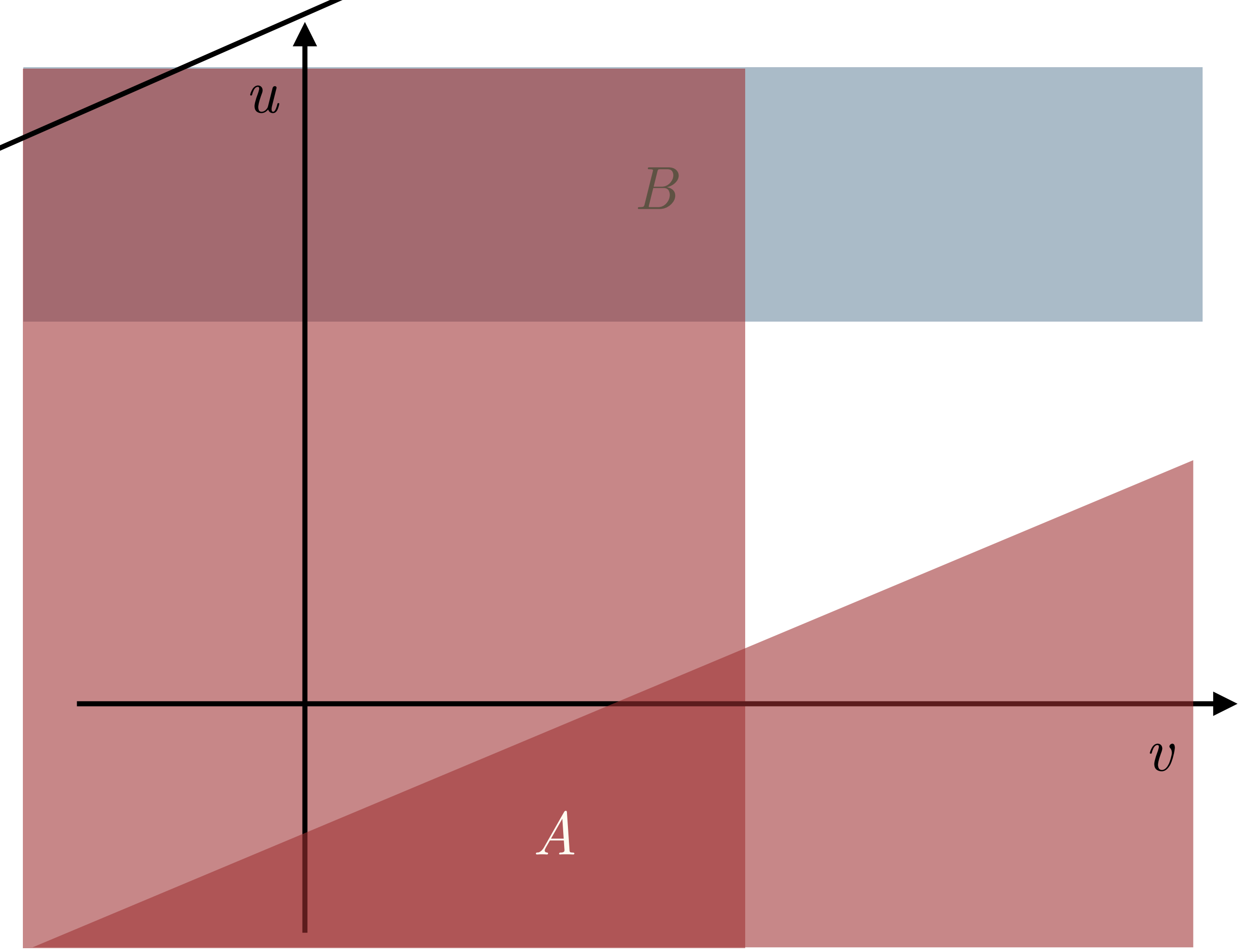

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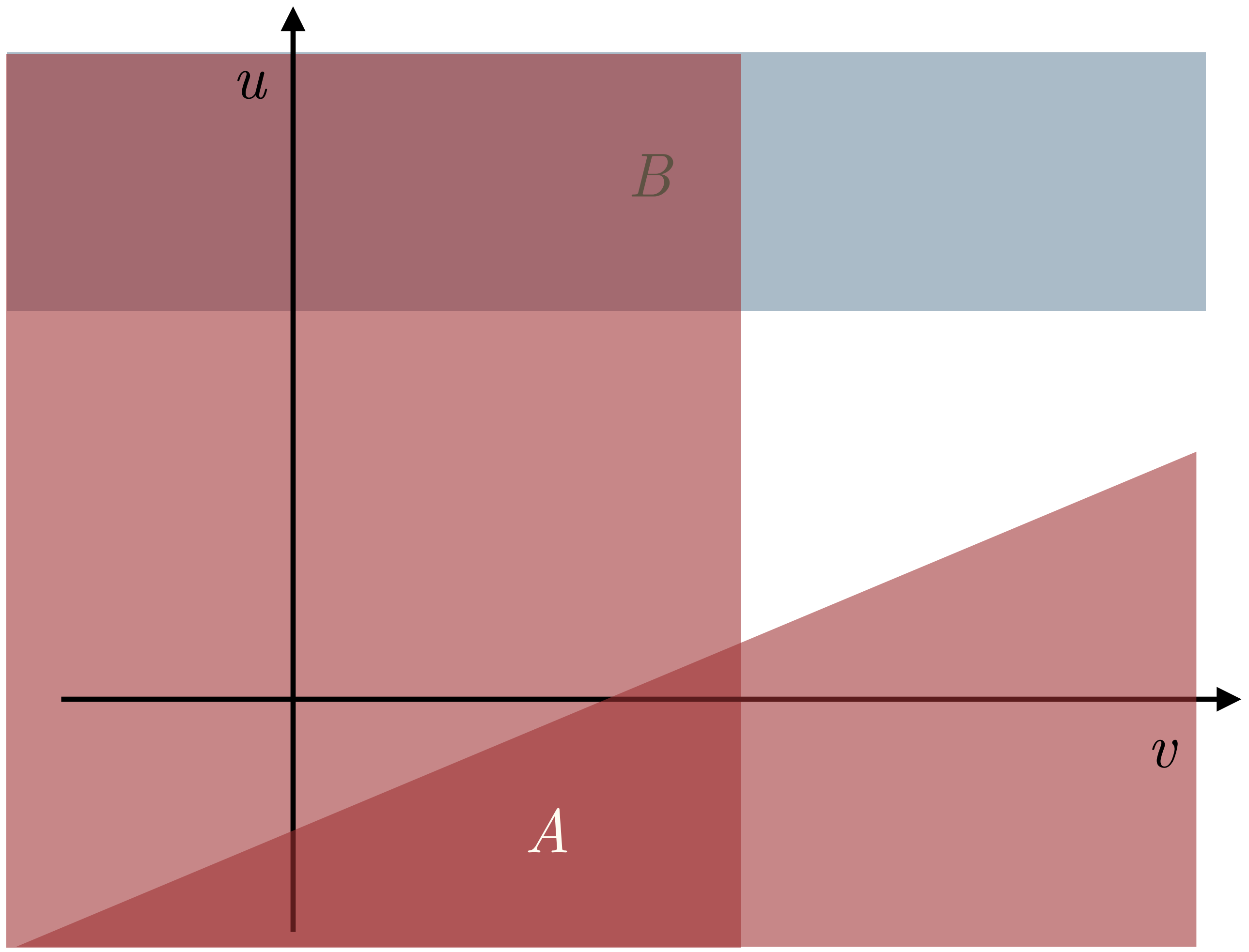

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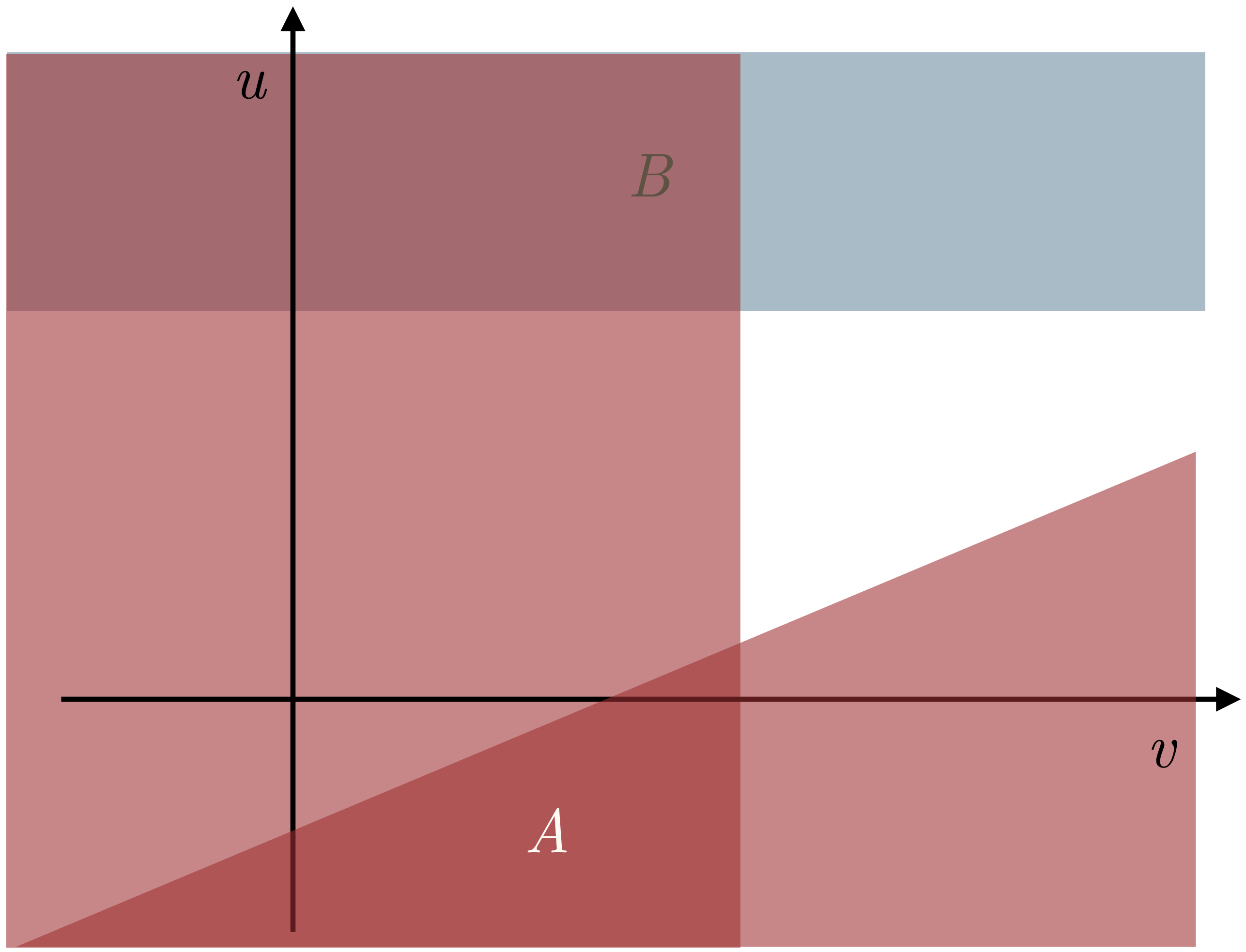


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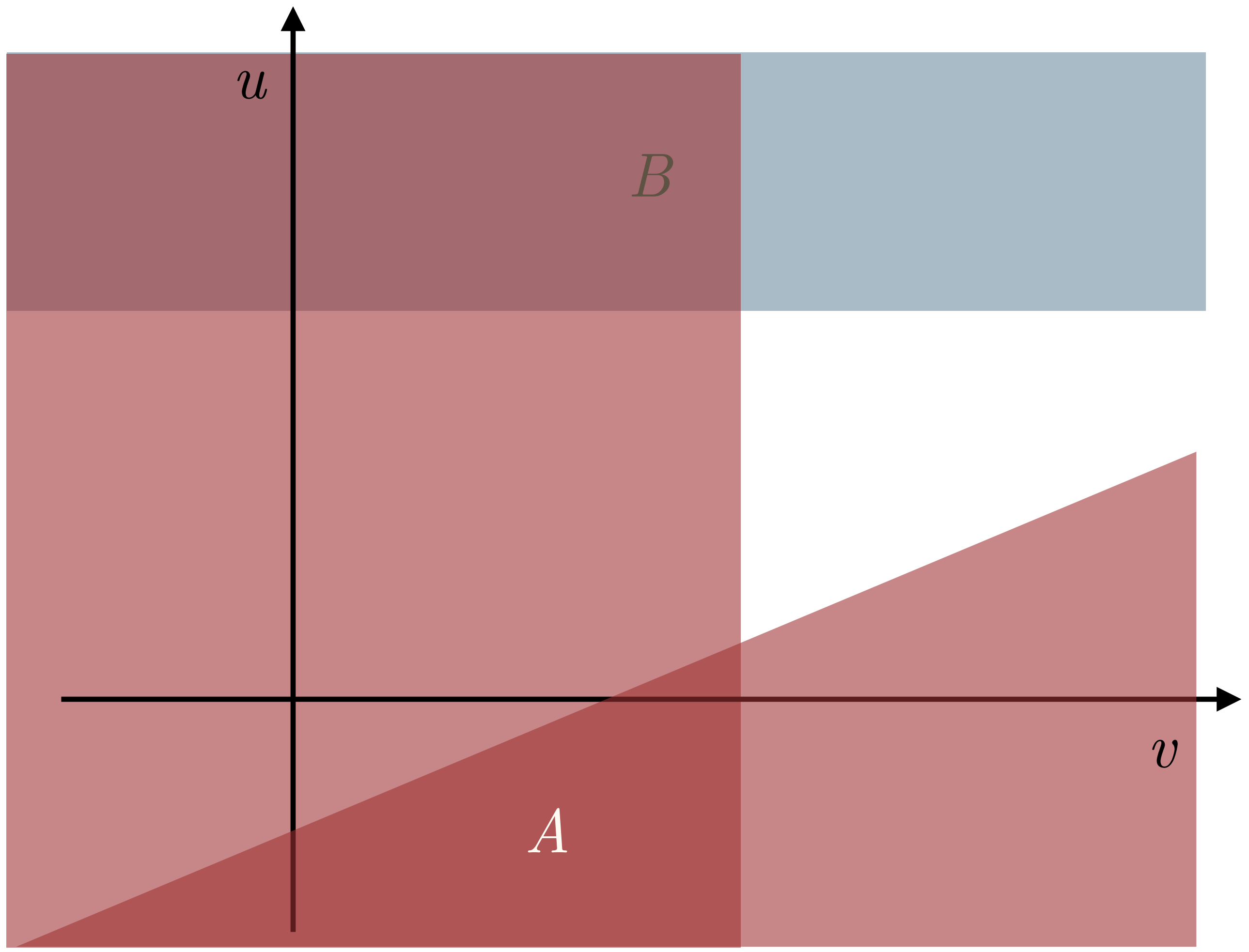
$0 <$ -$u - 1 + 1.5$



# LRA Interpolation

Assume that the expression bound that could not be satisfied was $1 > 0.5v - u$

and the bounds for the variables $u,\ v$ were

$u > 3$

$v < 3$

Assume that $(u > 3) \in B$ and $(v < 3) \in A,\ (1 > 0.5v - u) \in A.$

The interpolant for $A$ is obtained by summing to the expression the bounds in $A$ multiplied by their factors in the expression:

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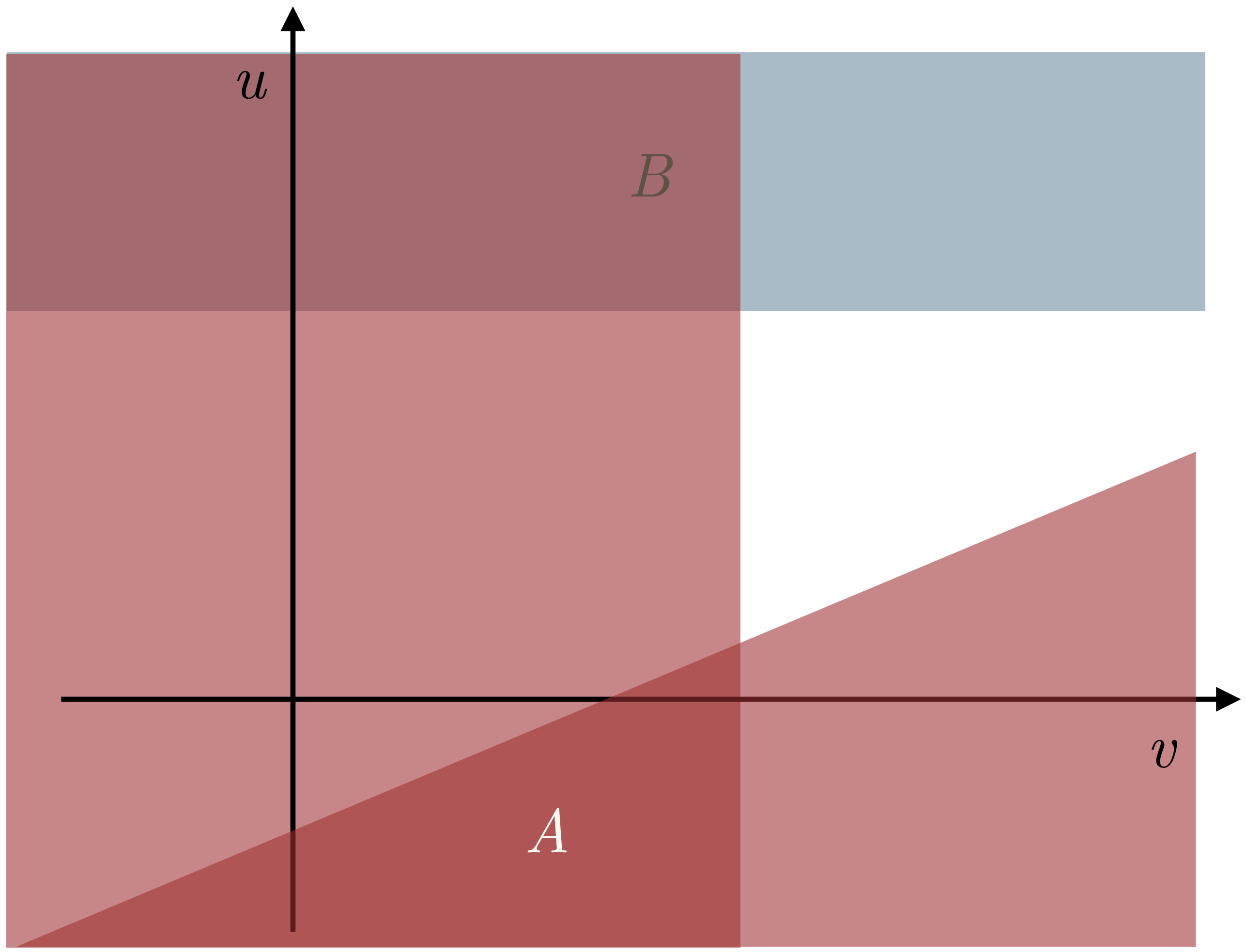

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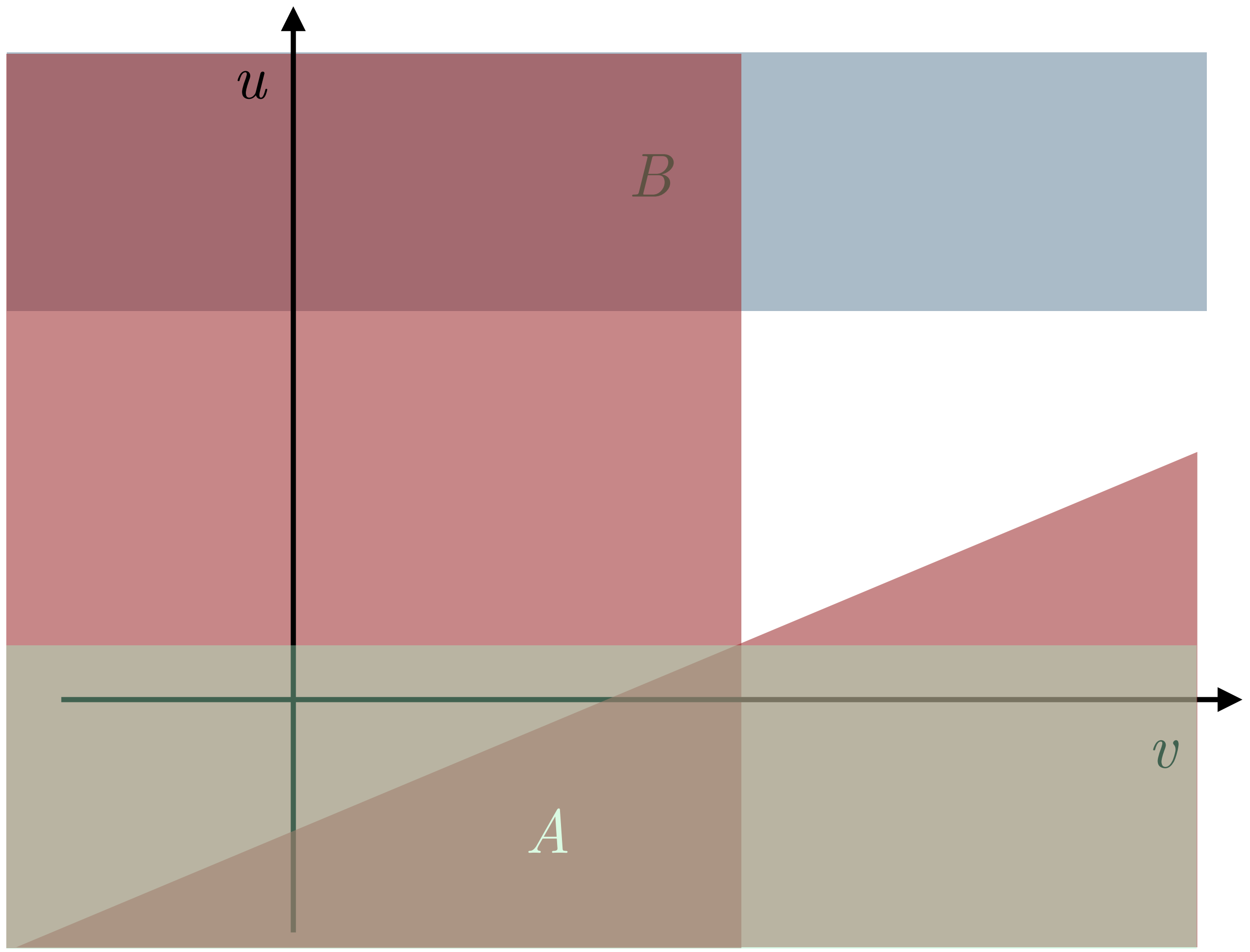

# Duality-based Interpolation for LRA

Given a primal interpolant

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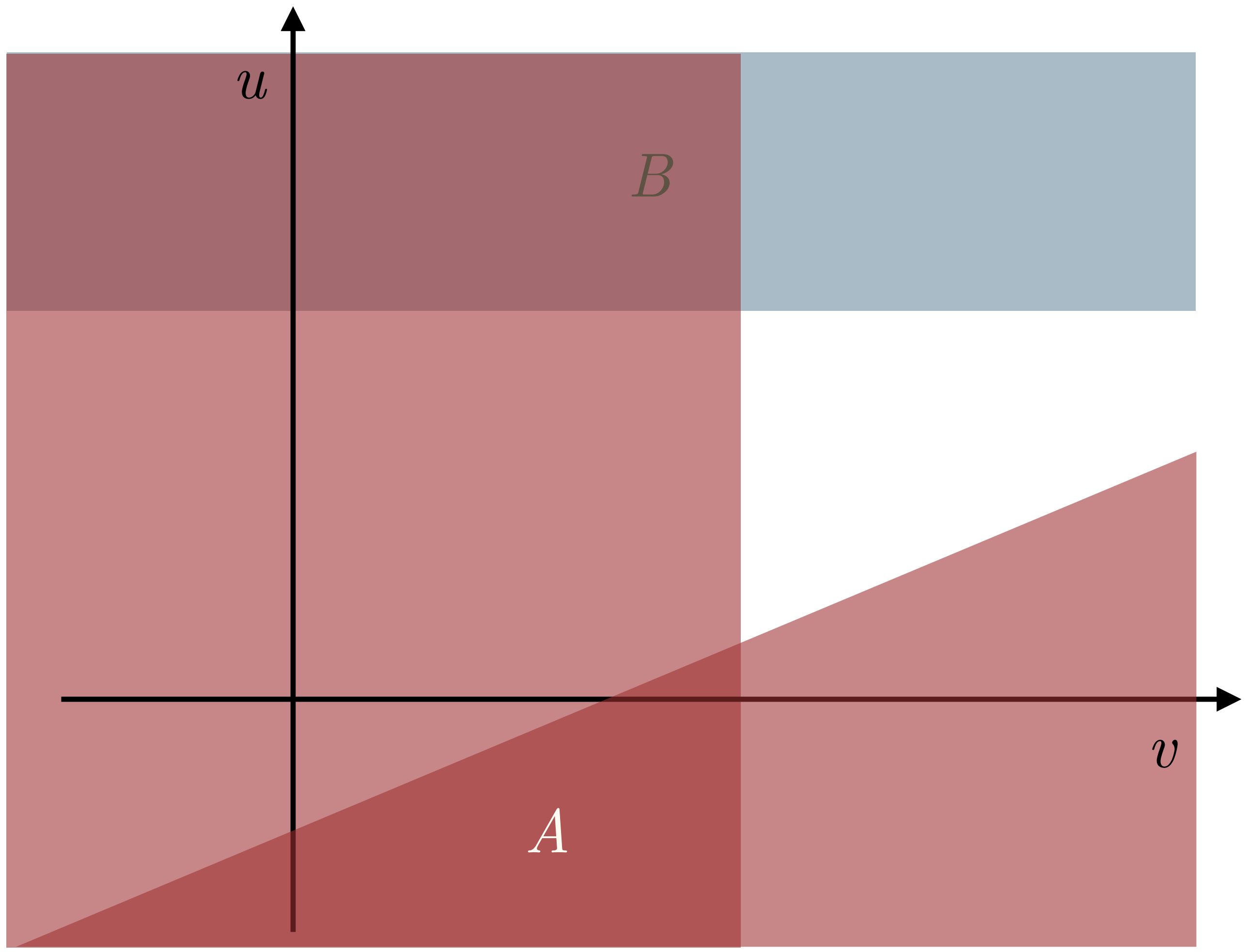

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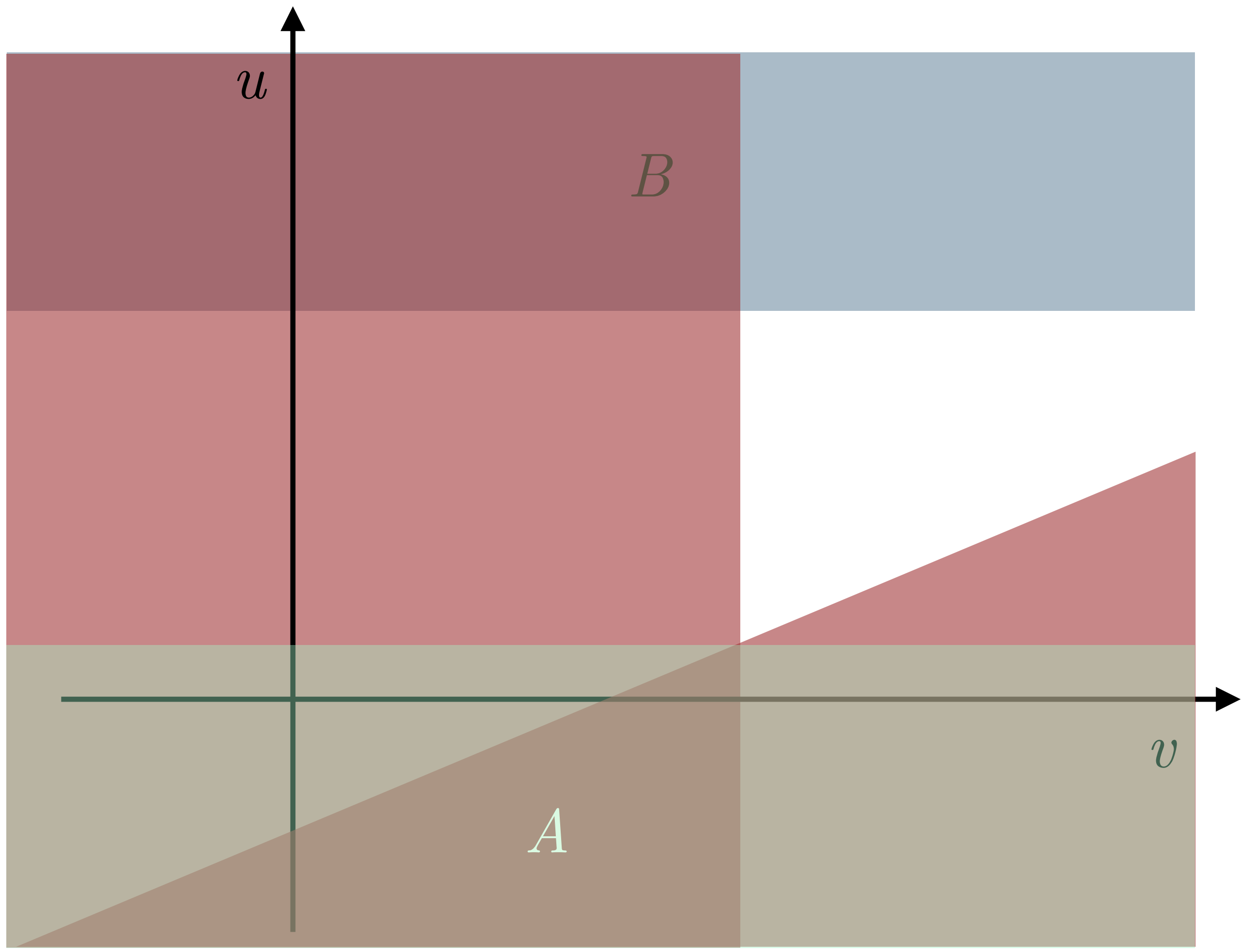

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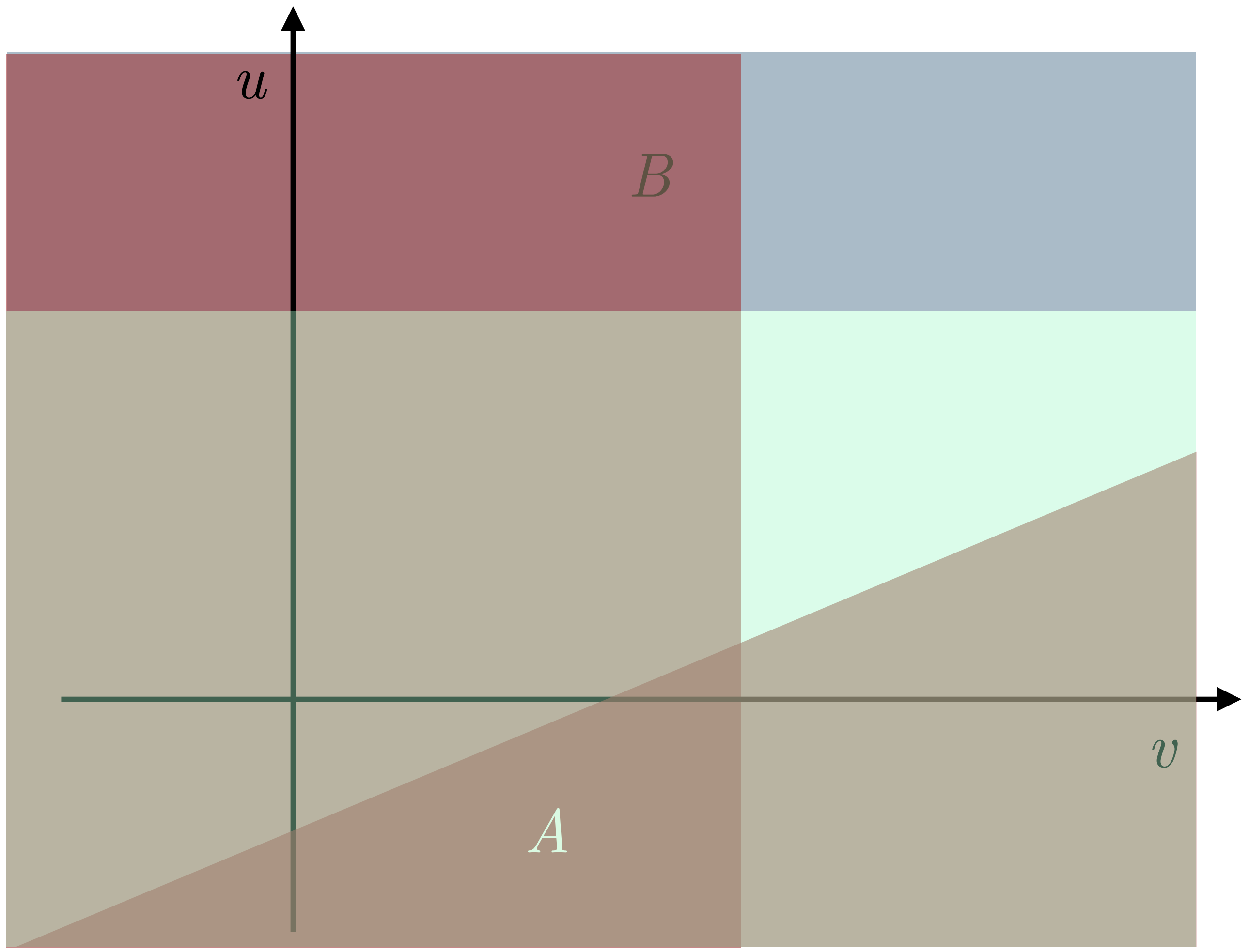

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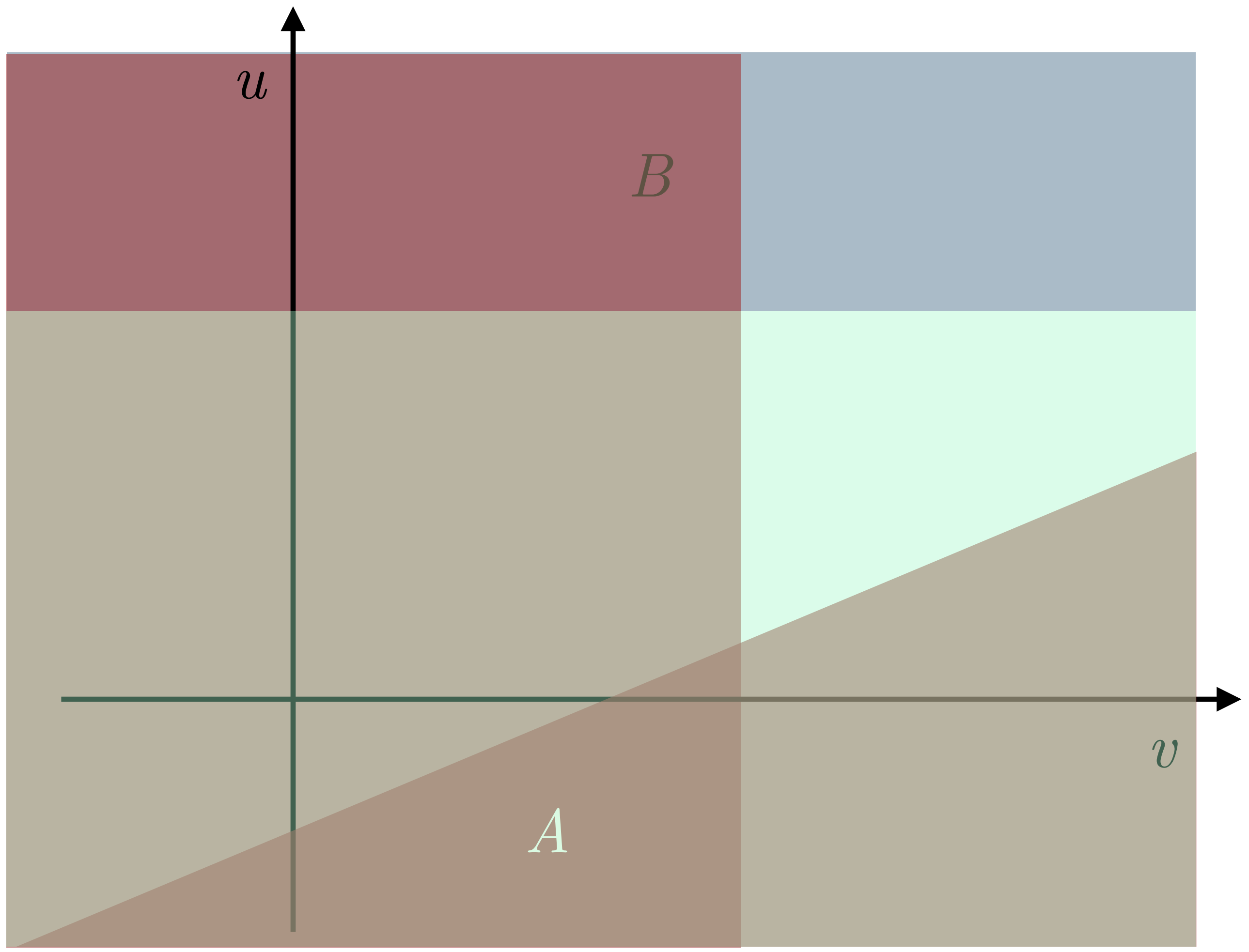

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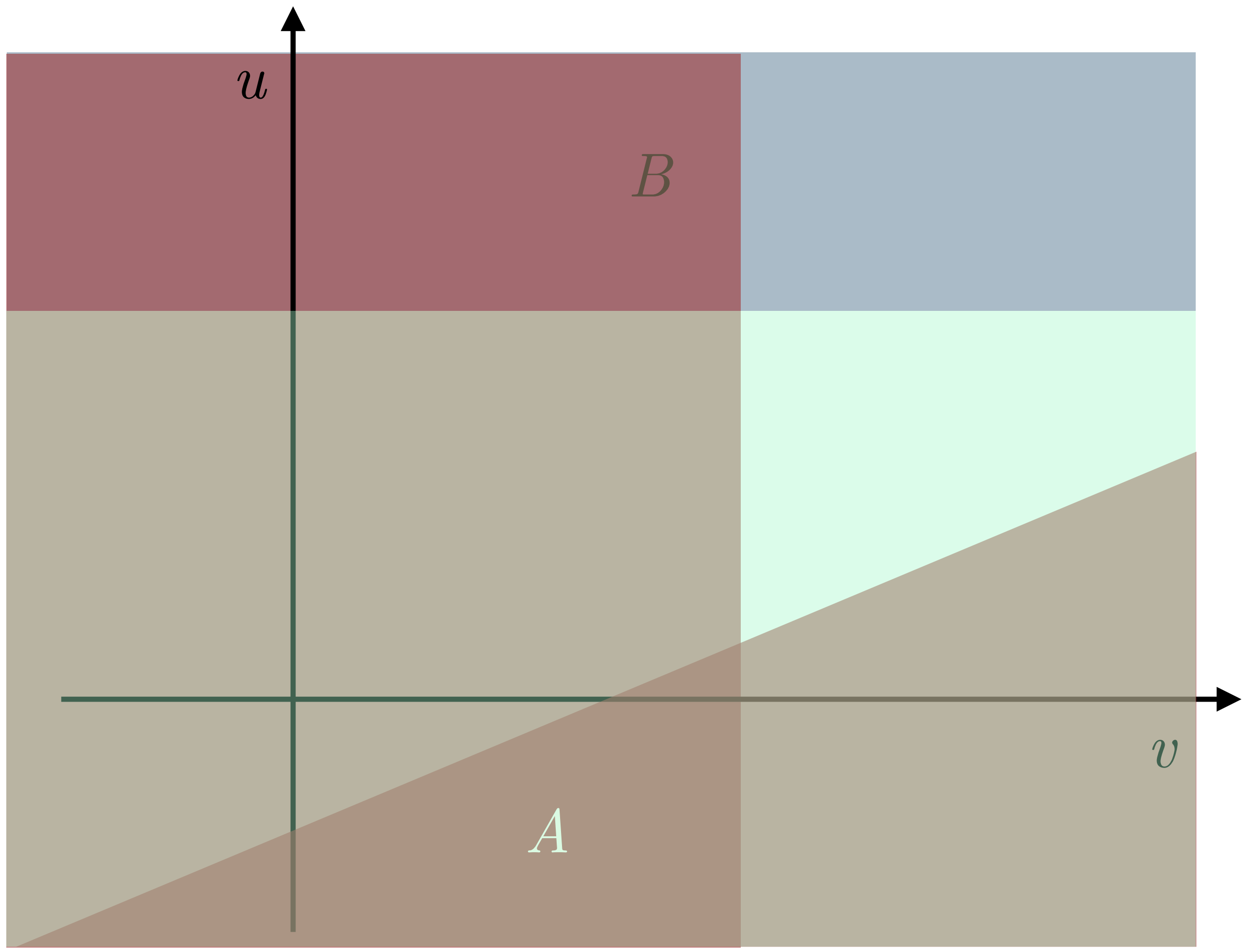

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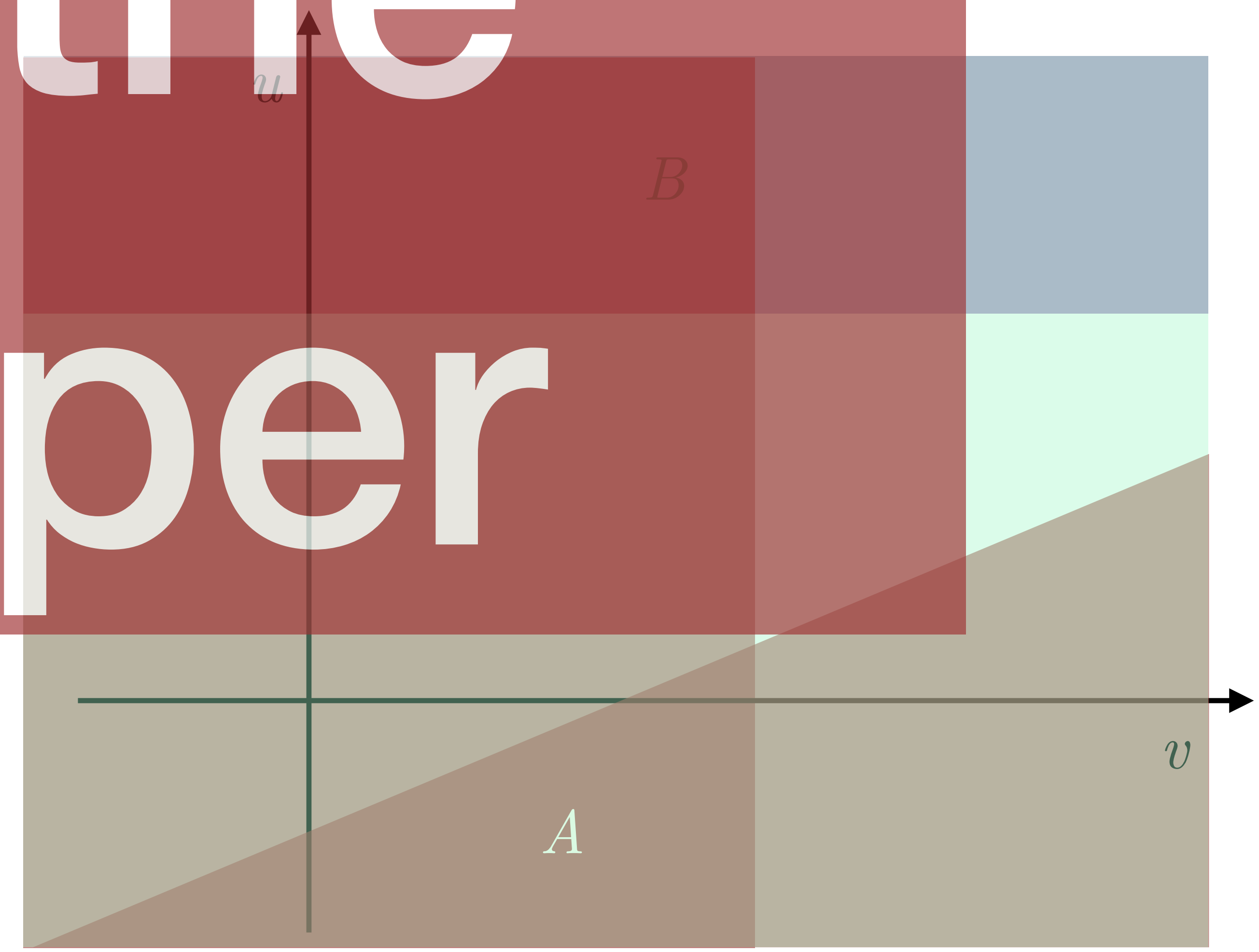
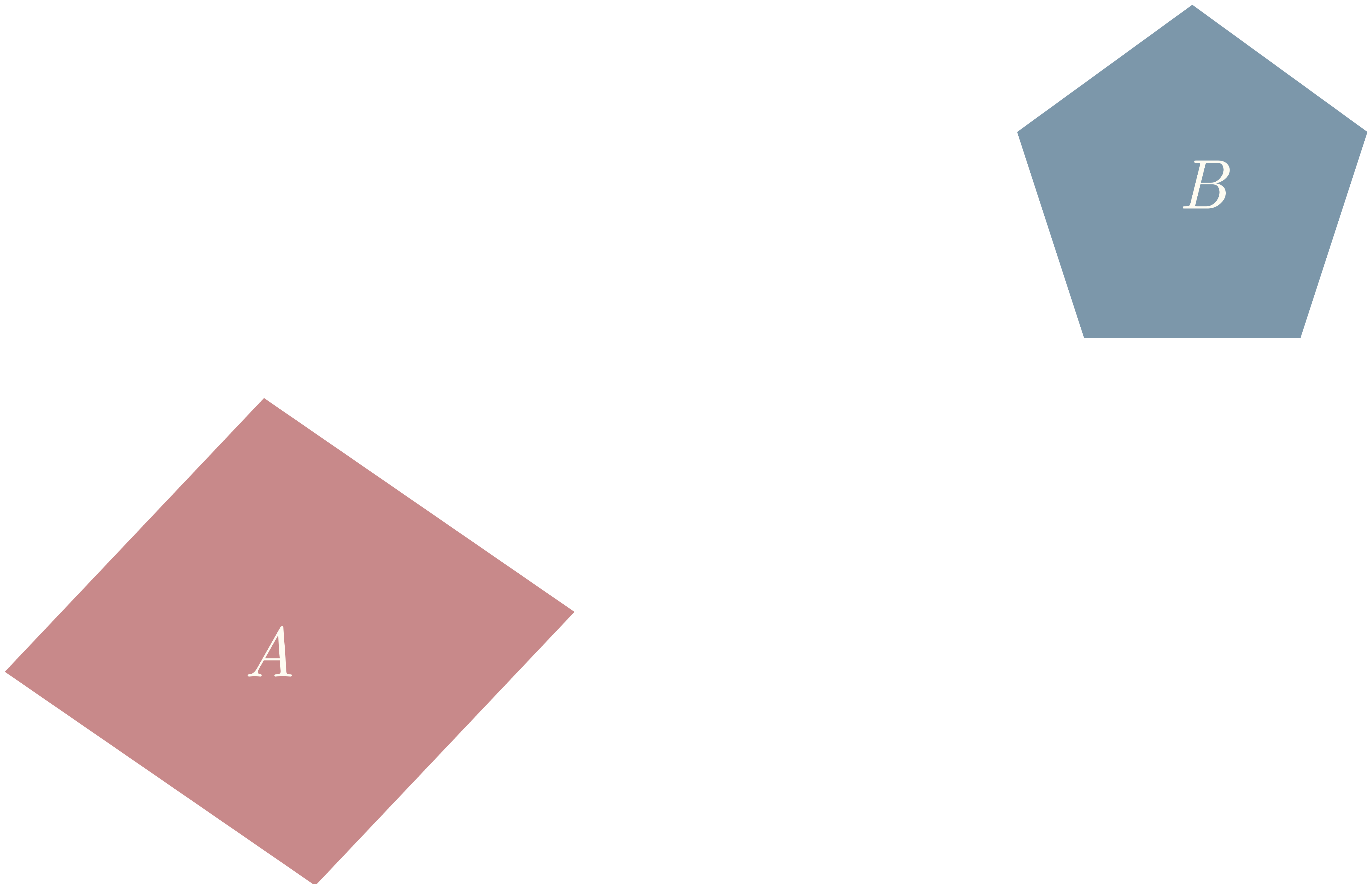
In the paper



# Interpolant Duality Visualized



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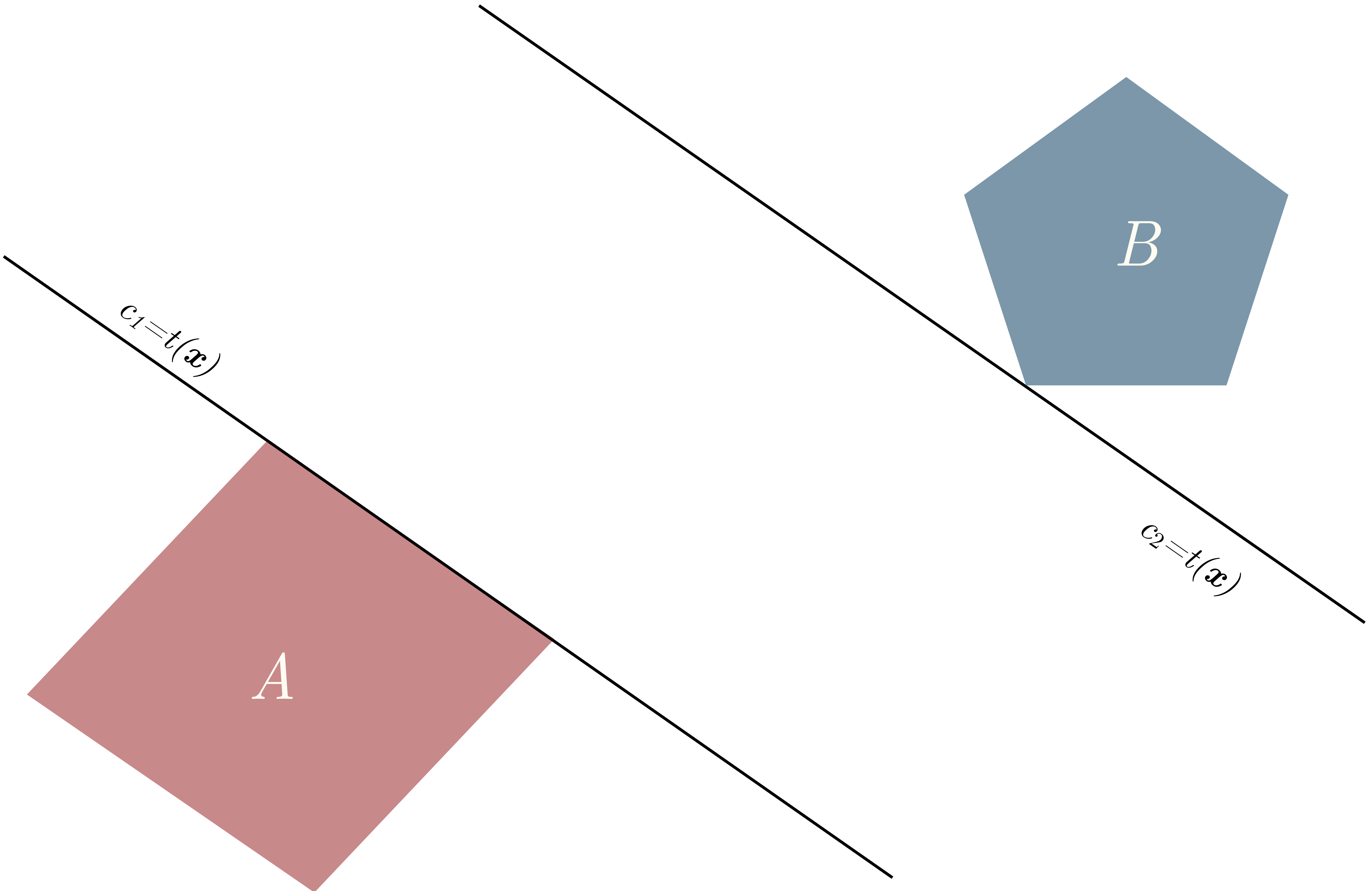

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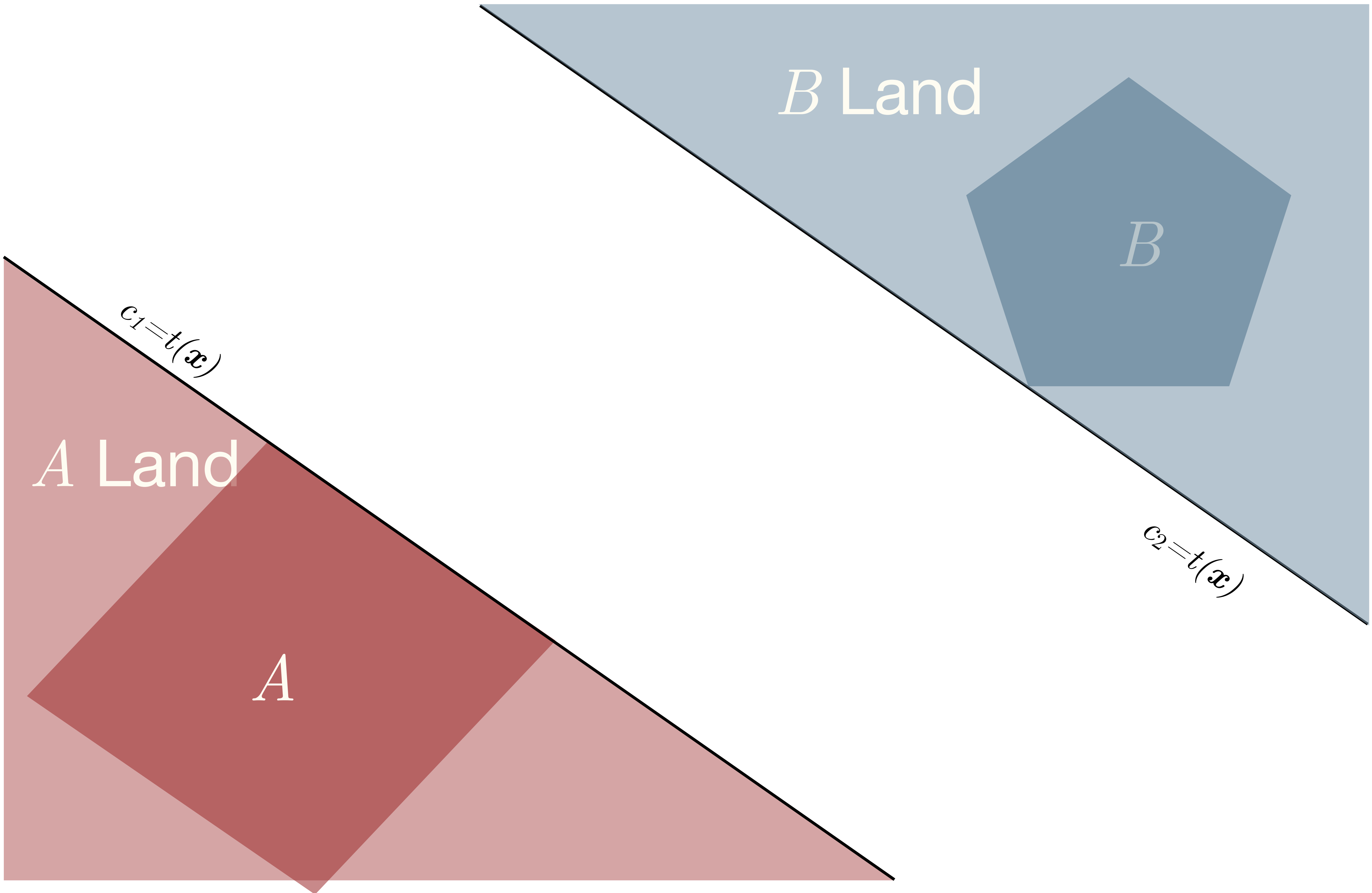

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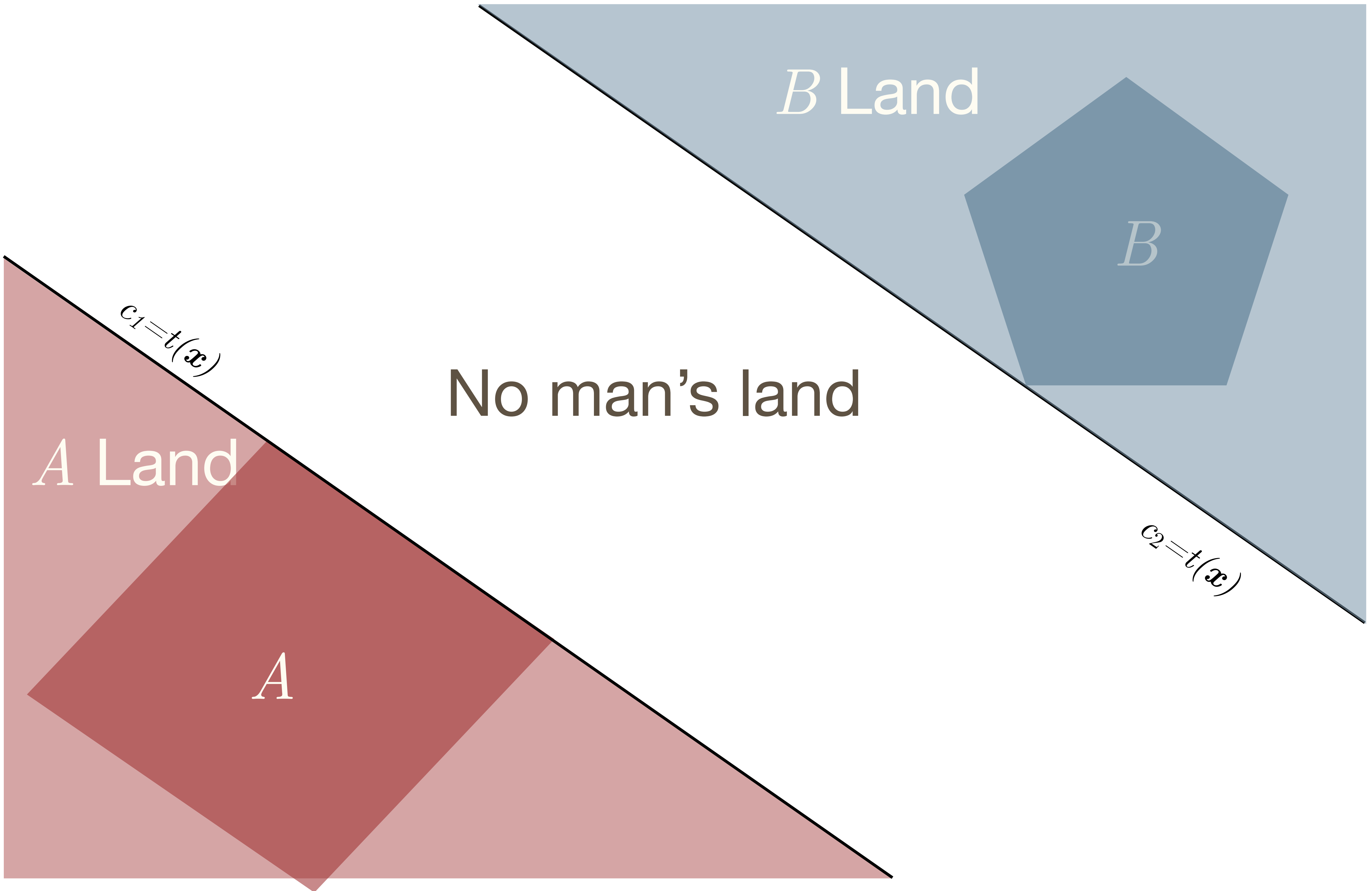

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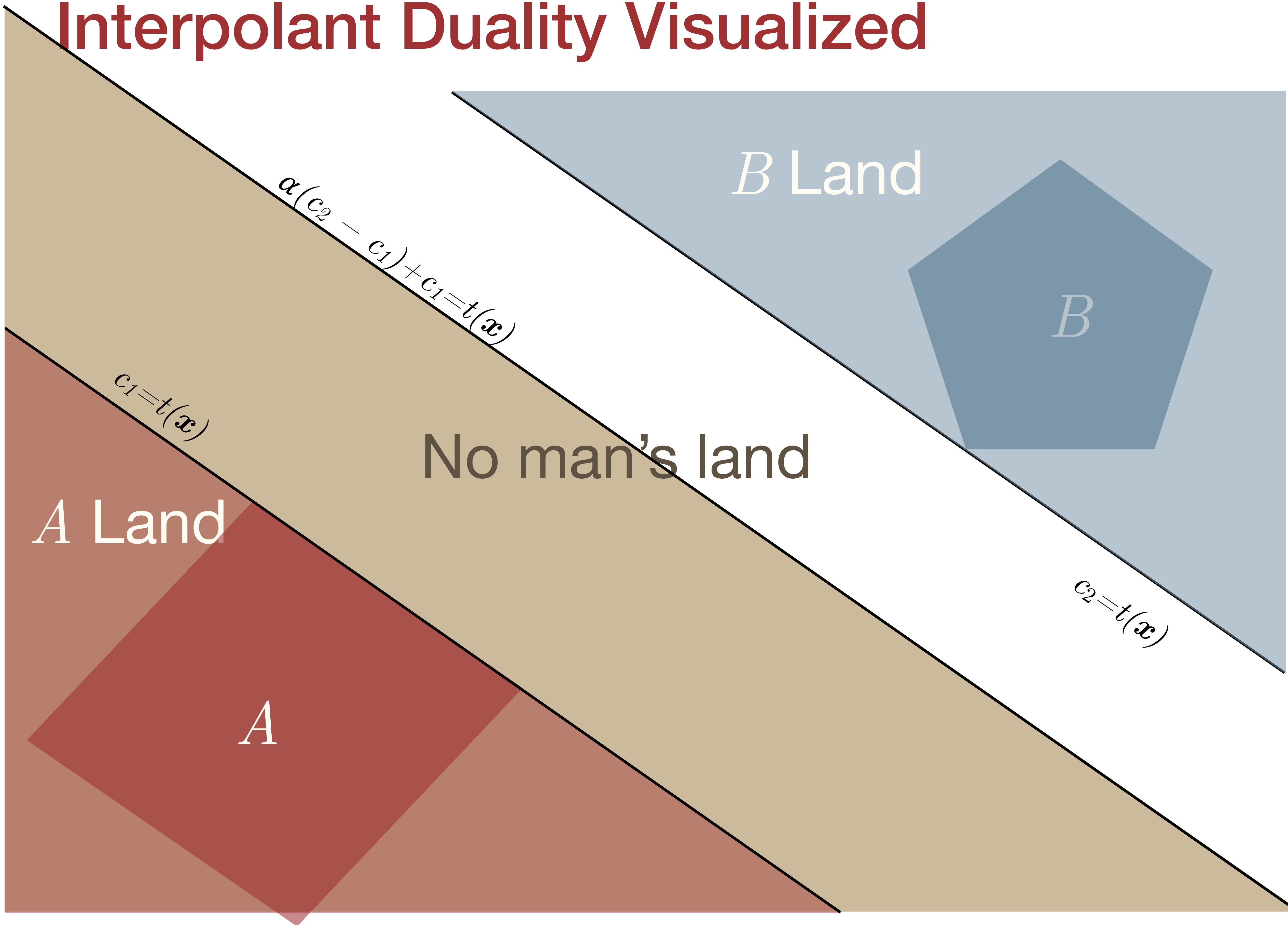

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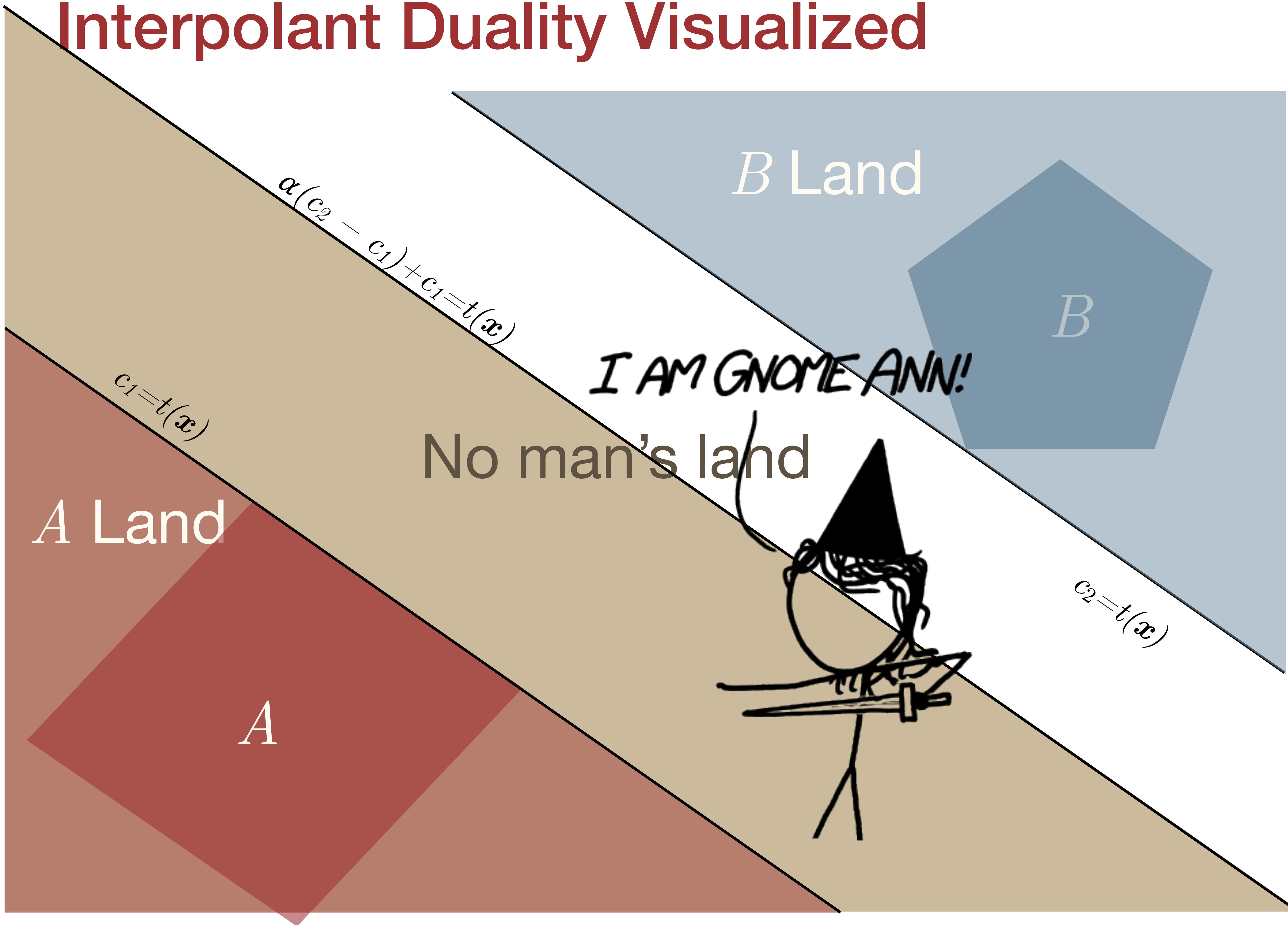
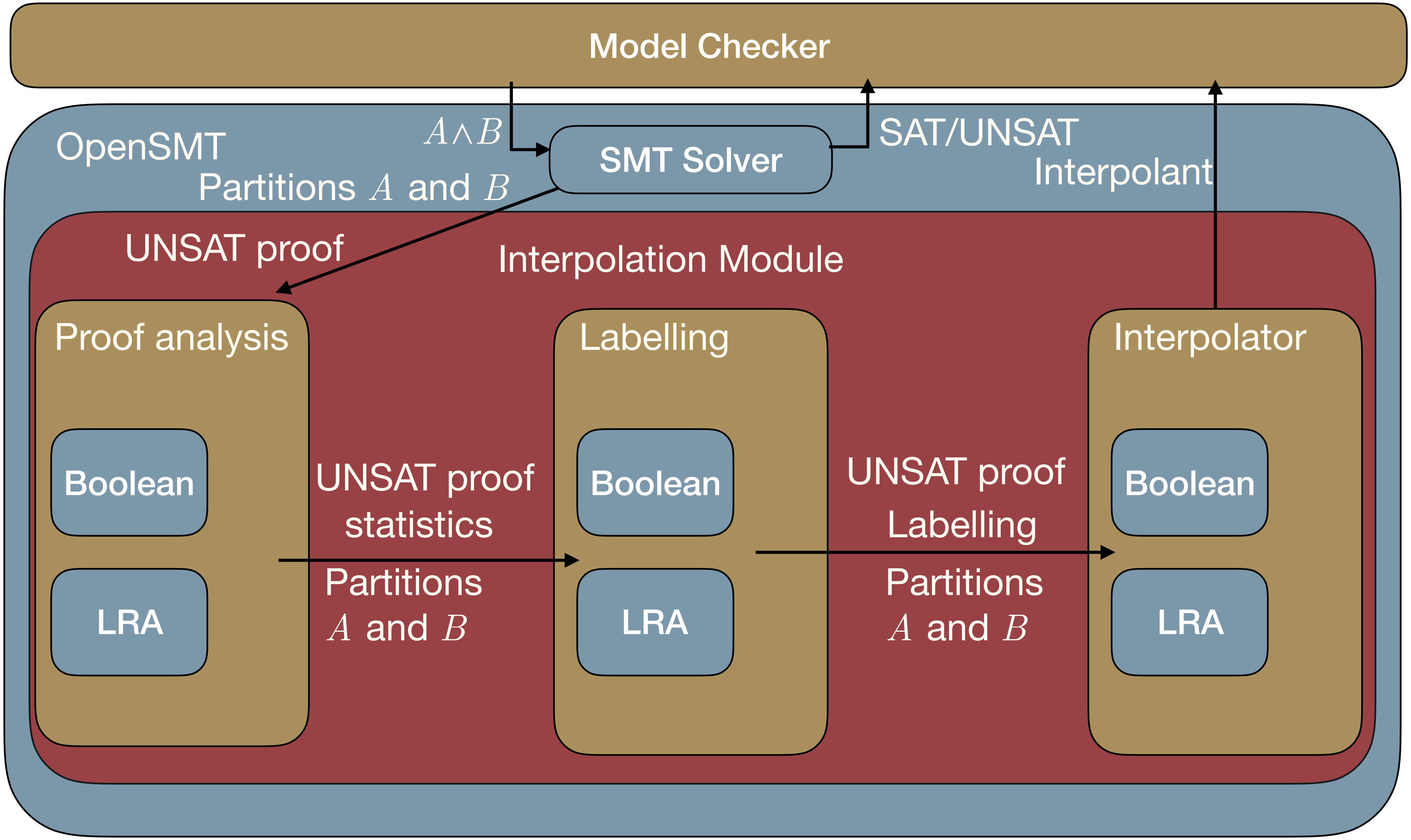
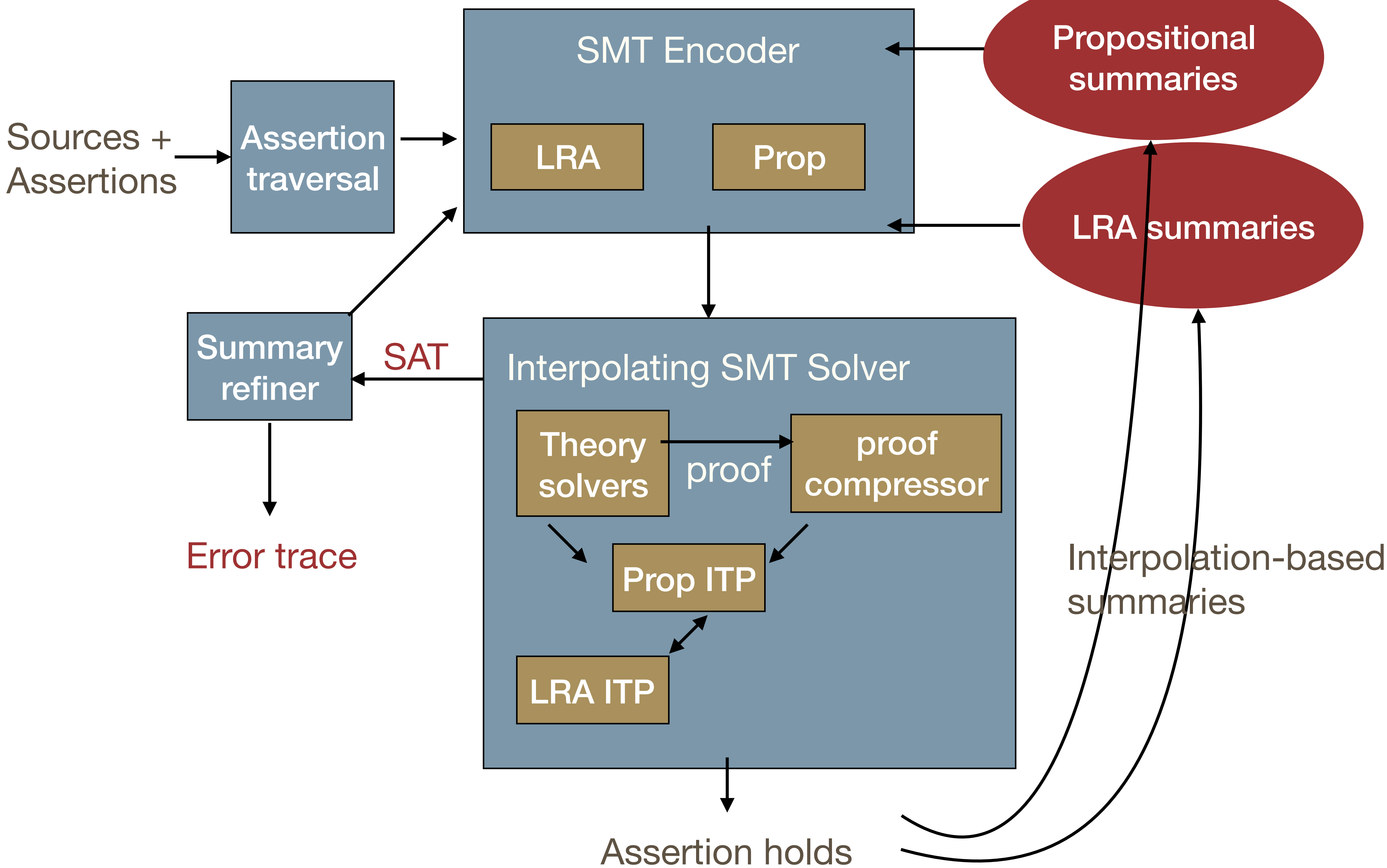


# Experiments on SV-COMP and HiFrog

# The Architecture Overview



# Implemented in HiFrog



# Results on SMT-LIB

Experiments with three LRA labelling functions:

Strong: the primal interpolant

Weak: the dual interpolant

$c = 0.5$: the interpolant between dual and primal

# Experiments on HiFrog

| ITP | floppy1 | kbfiltr1 | diskperf1 | mem | disk | Σ |
|---|---|---|---|---|---|---|
| **Strong** | 27100 | 5120 | 39900 | 25600 | 47600 | 145000 |
| $c = 0.5$ | 25100 | 5120 | 39200 | 25100 | 41500 | 136000 |
| **Weak** | 24800 | 5380 | 39200 | 25600 | 64000 | 159000 |

Number of HiFrog refinements (fixed propositional ITP algorithm)

- The difference between minimum and maximum is ~ 15%
- The $c = 0.5$ ITP provides the best results

# Related Work


Nikolaj Bjørner, Arie Gurfinkel:
*Property Directed Polyhedral Abstraction.* VMCAI 2015

Pudlák:
*Lower bounds for resolution and cutting plane proofs and monotone computations*. Journal of Symbolic Logic 1997.

McMillan:
*An Interpolating Theorem Prover*. Theoretical Computer Science 2005.

D'Silva, Kroening, Purandare, and Weissenbacher:
*Interpolant Strength*. VMCAI 2010.

Albarghouthi, McMillan:
*Beautiful interpolants. CAV 2013.*

Dutertre, de Moura:
*A fast linear-arithmetic solver for DPLL(T)*. Logical Methods in Computer Science 2012.

Alt, Hyvärinen, Asadi, and Sharygina:
*Duality-Based Interpolation for Quantifier-Free Equalities and Uninterpreted Functions. FMCAD 2017.*

# Conclusions

LRA interpolation with controlled strength

Provides an infinite family of interpolants based on interpolation duality

Integrated into a model checker

# Future work

Better heuristics for the labelling function

Apply to fix-point computations in other MC applications

Implementations available at

http://verify.inf.usi.ch/hifrog,

http://verify.inf.usi.ch/opensmt